Journal of Nonlinear Modeling and Analysis
Volume 5, Number 3, September 2023, 446–455
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2023.446

Iterative Positive Solutions to Nonlinear q-Fractional Differential Equations with Integral Boundary Value Conditions*

Yaqiong Cui[1,†], Shugui Kang[1] and Huiqin Chen[1]

Abstract This paper is concerned with the existence of positive solutions to nonlinear q-fractional differential equations yielding to the integral boundary value conditions. Under sufficient conditions of the nonlinearity, by using some iterative techniques, we get that this problem has two positive solutions and a unique positive solution respectively. Our results improve some recent work.

Keywords Integral boundary value condition, q-fractional differential equation, iterative technique, positive solutions

MSC(2010) 39A13, 31A10, 74H20.

1. Introduction

The purpose of this paper is to obtain the existence of positive solutions to the following integral boundary value problem of the q-fractional differential equation (BVP)

$$\begin{cases} (D_q^\alpha u)(t) + f(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ D_q^i u(0) = 0, \quad 0 \leqslant i \leqslant n-2, \quad u(1) = \lambda \int_0^1 u(s) \mathrm{d}_q s, \end{cases} \tag{1.1}$$

where $\alpha \in (n-1, n]$, $n \geqslant 3$ is an integer, $f \in C([0,1] \times [0, +\infty) \times (0, +\infty), (0, +\infty))$, $\lambda \in (0, [\alpha]_q)$, and D_q^α denotes the α order fractional q-derivative operator of the Riemann-Liouville type.

Fractional q-difference equations have received considerable attention due to their ability to accurately describe various phenomena such as mathematical models, quantum calculus and engineering problems. More recently, various fractional q-difference systems have been reduced to the search of solutions by iterative methods [2, 8, 10, 12] and by other fixed point theories [7, 14, 15]. For example, by using the monotone iterative methods, the authors [8, 12] dealt with the existence of positive solutions to fractional q-difference equations, and among these results, f must be a monotonic function with respect to the only spatial variable. Especially, by iterative algorithm, Mao, Zhao and Wang [10] gained the unique positive solution to the

† The corresponding author.
Email address: cuiyaqiongdt@163.com (Y. Cui), dtkangshugui@126.com (S. Kang), dtdxchq@126.com (H. Chen)
[1] School of Mathematics and Statistics, Shanxi Datong University, Datong, Shanxi 037009, China
* The authors were supported by National Natural Science Foundation of China (No. 11871314) and Natural Science Foundation of Datong City (No. 2020147).

fractional q-difference Schrödinger equation whose nonlinear term has two spatial variables, and the initial value of iterative series is a constant multiple of a point in the cone but not a fixed upper and lower solutions as [12]. Integral boundary value problems of fractional difference systems may respond to some special features such as the blood flow, chemical engineering and other issues. Therefore, much interest has been given to study the problems in this area (see [1,2,6,8,9,17]). In fact, the authors [2,8] have supplied a general form with respect to this kind of fractional q-difference boundary value problems. Moreover, in [2], Cui, Kang and Chen have given the corresponding expression to the Green's function. By carefully analyzing these works of [1,8,10,12,14,16–18], the authors have found that those results in [8,10,12,18] can be effectively promoted.

During our discussion, an inverse symmetry subset in the cone plays a fundamental role, which skillfully converts the existence of a positive solution of BVP (1.1) into the existence of a fixed point for the equivalent integral operator in this set. Compared with the results studied recently, these results presented here have improved in some aspects. First, here the fractional order $\alpha > 2$ and the nonlinear term f is mixed monotone, thus it includes much more types of functions. Second, we extend the ideas of [18] to establish richer conditions on f. In particular, when comparing with the preceding proof in Theorem 3.1 of [18], one can see that what we actually do reveals the characteristics of solutions for the similar conditions on the nonlinearity term f.

The main conditions on f are as follows:

(H) For each fixed $t \in (0,1)$, $f(t,u,v)$ is increasing on u and decreasing on v and

$$0 < \int_0^1 (1-qs)^{(\alpha-1)} f(s, s^{\alpha-1}, s^{\alpha-1}) \mathrm{d}_q s < +\infty.$$

(H_1) There exist constants $\sigma \in (0,1)$, $\rho \in (0,1]$ such that for all $(t,u,v) \in [0,1] \times [0,+\infty) \times (0,+\infty)$,

$$f(t, ru, r^{-1}v) \geqslant r[1+\rho(r^{-\sigma}-1)] f(t,u,v), \quad r \in (0,1]. \tag{1.2}$$

(H_2) There exists a constant $\sigma \in (0,1)$ such that for all $(t,u,v) \in [0,1] \times [0,+\infty) \times (0,+\infty)$,

$$f(t, ru, r^{-1}v) \geqslant r^{\sigma} f(t,u,v), \quad r \in (0,1]. \tag{1.3}$$

(H_3) There exist constants λ_i, $\mu_i (i=1,2)$ satisfying $0 < \lambda_1 \leqslant \lambda_2 < 1$, $0 < \mu_1 \leqslant \mu_2 < 1$ and $\lambda_2 + \mu_2 < 1$ such that for all $(t,u,v) \in [0,1] \times [0,+\infty) \times (0,+\infty)$,

$$r^{\lambda_2} f(t,u,v) \leqslant f(t,ru,v) \leqslant r^{\lambda_1} f(t,u,v), \quad r \in (0,1]; \tag{1.4}$$

$$r^{\mu_2} f(t,u,v) \leqslant f(t,u,r^{-1}v) \leqslant r^{\mu_1} f(t,u,v), \quad r \in (0,1]. \tag{1.5}$$

Remark 1.1. For $r \in (1,+\infty)$, the condition (H_1) implies

$$f(t, ru, r^{-1}v) \leqslant r[1+\rho(r^{\sigma}-1)]^{-1} f(t,u,v), \tag{1.6}$$

and the condition (H_2) implies

$$f(t, ru, r^{-1}v) \leqslant r^{\sigma} f(t,u,v). \tag{1.7}$$

.