Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 5, Number 3, September 2023, 456–470 DOI:10.12150/jnma.2023.456

Dynamical Analysis for a General Jerky Equation with Random Excitation*

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Abstract A general jerky equation with random excitation is investigated in this paper. Before introducing the random excitation term, the equation is reduced to a two-dimensional model when undergoing a Hopf bifurcation. Then the model with the parametric excitation and external excitation is converted to a stochastic differential equation with singularity based on the stochastic average theory. For the equation, its dynamical behaviors are analyzed in different parameters' spaces, including the stability, stochastic bifurcation and stationary solution. Besides, numerical simulations are given to show the asymptotic behavior of the stationary solution.

Keywords Jerky equation, stochastic stability, stochastic bifurcation, stationary solution

MSC(2010) 34D20, 34F05, 34F10.

1. Introduction

In the real world, the motion of objects is inevitably influenced by environmental factors, internal structures and other unknown elements. As a result, stochastic systems can predict the evolution of trends more precisely. Furthermore, it fosters the development of random dynamical systems [2] that have widespread applications in physics [15, 24, 25], economics [4, 8, 12] and ecosystems [5, 9, 10, 13, 17, 26].

The jerky equation, which is a third-order explicit autonomous ordinary differential equation represented as $\dddot{u} = J(u, \dot{u}, \ddot{u})$, describes the motion of objects in terms of displacement u, velocity $\dot{u}$, acceleration $\ddot{u}$ and jerk $\dddot{u}$. In 1998, Eichhorn et al., proposed seven jerky equations $JD_1 - JD_7$, which encompassed nineteen important physical chaotic frameworks (A-S) [6] and Rössler's toroidal (TR) model [21]. Later on, Ren, Yu and Zhu, [20] performed a comprehensive dynamical analysis of discrete-time JD_1 and continuous-time JD_1 with delayed feedback. Correspondingly, Tang, Zhang and Ren [22] systematically investigated the following general jerky equation that comprises $JD_1 - JD_7$

$$\dddot{u} = \alpha_0 + \alpha_1 u + \alpha_2 \dot{u} + \alpha_3 \ddot{u} + \alpha_4 u^2 + \alpha_5 \dot{u}^2 + \alpha_6 u\dot{u} + \alpha_7 u\ddot{u}, \tag{1.1}$$

where α_i are the parameters, and $i = 0, 1, \ldots, 7$. They determined precise bifurcation conditions for Fold, Hopf, Zero-Hopf and Bogdanov-Takens bifurcations. The

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*The authors were supported by the National Natural Science Foundation of China (Grant No. 52071298).

rich dynamical behaviors of equation (1.1) appeal us to investigate its stochastic dynamics, when it is disturbed by the parametric and external excitations. Therefore, we introduce a new stochastic model by incorporating noises into equation (1.1). Before adding the stochasticity, we reduce (1.1) to a two-dimensional equation, when it undergoes Hopf bifurcation by the center manifold theory. Then we add the parametric and external excitations to the two-dimensional equation, and transform it into a stochastic differential equation (SDE) with singularity by using the Khasminskii limit theorem [11,23] and the stochastic averaging method [16,19]. Interestingly, we obtain a nonlinear SDE comprising a singularity term. Following that, we discuss the stochastic stability using the singular boundary theory [14,27], and prove that the SDE without singularity undergoes the stochastic D-bifurcation and stochastic P-bifurcation [3,7,18]. Furthermore, we calculate the stationary solution for SDE with singularity by deriving its probability density function. Finally, we give numerical simulations to show the asymptotic behavior of the stationary solution with respect to various parameters.

2. Preparation

In this section, we reduce equation (1.1) to a two-dimensional system, when it undergoes Hopf bifurcation.

By setting $\dot{u} = v, \dot{v} = w$ in (1.1), the equilibrium $(u^*, 0, 0)$ where Hopf bifurcation occurs in [22] is as follows.

- $u^* = -\frac{\alpha_0}{\alpha_1}$, when $\alpha_1 \neq 0$, $\alpha_4 = 0$;
- $u^* = \frac{-\alpha_1 - \sqrt{\Delta}}{2\alpha_4}$ or $u^* = \frac{-\alpha_1 + \sqrt{\Delta}}{2\alpha_4}$, when $\alpha_4 \neq 0$, $\alpha_1^2 > 4\alpha_4\alpha_0$, where $\Delta = \sqrt{\alpha_1^2 - 4\alpha_4\alpha_0}$.

Making the transformation $\bar{u} \to u - u^*, \bar{v} \to v, \bar{w} \to w$, and still using the original notations u, v, w, system (1.1) becomes

$$\begin{cases} \dot{u} = v, \\ \dot{v} = w, \\ \dot{w} = \alpha_4 u^2 + \alpha_5 v^2 + \alpha_6 uv + \alpha_7 uw \\ \qquad +(2\alpha_4 u^* + \alpha_1)u + (\alpha_6 u^* + \alpha_2)v + (\alpha_7 u^* + \alpha_3)w. \end{cases} \tag{2.1}$$

The Jacobian matrix of (2.1) evaluated at $(0, 0, 0)$ is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma & -\beta & -\alpha \end{pmatrix}. \tag{2.2}$$

The characteristic equation of (2.1) at the equilibrium $(0, 0, 0)$ takes the form $\lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma = 0$, where $\alpha = -(\alpha_7 u^* + \alpha_3)$, $\beta = -(\alpha_6 u^* + \alpha_2)$, and $\gamma = -(2\alpha_4 u^* + \alpha_1)$. Substituting $\lambda = i\mu$ into the characteristic equation yields a relation among α, β and γ. If

$$\beta = \frac{\gamma}{\alpha}, \quad \beta = \mu^2,$$