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Bifurcations and Exact Solutions of the Gerdjikov-Ivanov Equation*

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Abstract For the Gerdjikov-Ivanov equation, by using the method of dynamical system, this paper investigates the exact explicit solutions with the form $q(x,t) = \phi(\xi)\exp\left[i(\kappa x - \omega t + \theta(\xi))\right], \xi = x - ct$. In the given parameter regions, more than 14 explicit exact parametric representations are presented.

Keywords Bifurcation, exact solution, planar Hamiltonian system, Gerdjikov-Ivanov equation

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1. Introduction

In [8], Fang stated that "The nonlinear Schrödinger (NLS) equation is one of the most generic soliton equations, and arises from a wide variety of fields such as quantum field theory, weakly nonlinear dispersive water waves and nonlinear optics. To study the effect of higher-order perturbations, various modifications and generalizations of the NLS equations have been proposed and studied. Among them, there are three celebrated equations with derivative-type nonlinearities, which are called the derivative nonlinear Schrödinger (DNLS) equations. One is the Kaup-Newell equation [17]:

$$iq_t + q_{xx} + i(|q|^2 q)_x = 0, \tag{1.1}$$

which is usually called DNLSI. The second type is the Chen-Lee-Liu equation [4]:

$$iq_t + q_{xx} + i|q|^2 q_x = 0, \tag{1.2}$$

which is called DNLSII. The last one takes the form [13]:

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2}q^3(q^*)^2 = 0, \tag{1.3}$$

which is called the Gerjikov–Ivanov (GI) equation or DNLSIII. In equation (1.3), q^* denotes the complex conjugation of q.

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Equation (1.3) has been studied by many authors (see [1, 2, 5–7, 9–12, 14–16, 18, 21–32]). The purpose of this paper is to study some new exact traveling wave solutions in explicit form of the GI equation by using the bifurcation theory of dynamical system. We assume that the exact solutions of equation (1.3) take the form:

$$q(x,t) = \phi(\xi)\exp\left[i(\kappa x - \omega t + \theta(\xi))\right], \quad \xi = x - ct, \tag{1.4}$$

where c is the wave velocity, and $\phi(\xi), \theta(\xi)$ are two functions with variable ξ, κ and ω are two constant parameters. Substituting (1.4) into equation (1.3) and separating the real and imaginary parts respectively, we have

$$\phi'' + (c-2\kappa)\theta'\phi - (\kappa + \theta')\phi^3 + (\omega - \kappa^2)\phi - (\theta')^2\phi = 0, \tag{1.5}$$

$$\phi'\phi^2 - \theta''\phi - 2\phi'\theta' + (c-2\kappa)\phi' = 0, \tag{1.6}$$

where "$'$" is the derivative with respect to ξ. Integrating (1.6), it follows that $(c-2\kappa)\phi + \frac{1}{3}\phi^3 = C_1 + \theta'\phi + \int \theta' d\phi$, where C_1 is an integral constant. Thus, we obtain $C_1 = 0$ and

$$\theta' = \frac{1}{2}(c-2\kappa) + \frac{1}{4}\phi^2, \quad \theta(\xi) = \frac{1}{2}(c-2\kappa)\xi + \frac{1}{4}\int \phi^2(\xi)d\xi. \tag{1.7}$$

Substituting (1.7) into (1.5), we obtain the following planar dynamical system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\left(\left(\frac{1}{4}c^2 - c\kappa + \omega\right)\phi - \frac{1}{2}c\phi^3 + \frac{3}{16}\phi^5\right). \tag{1.8}$$

System (1.8) has the first integral:

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{1}{2}\alpha\phi^2 - \frac{c}{8}\phi^4 + \frac{1}{32}\phi^6 = h, \tag{1.9}$$

where $\alpha = \frac{1}{4}c^2 - c\kappa + \omega$.

System (1.8) is a three-parameter planar dynamical system depending on the parameter group (c, κ, ω). Since the parametric representations of the phase orbits defined by the vector fields of system (1.8) give rise to all exact solutions with the form (1.4) of equation (1.3), we need to investigate the bifurcations of phase portraits for system (1.8) in the $(\phi, y)-$phase plane as the parameters are changed (see [19, 20, 33]).

The main result in the present paper is summarized as follows.

Theorem 1.1. *Assume that the parameter $c > 0$ of system (1.8) is fixed. Consider the solutions of equation (1.3) with the form $q(x,t) = \phi(\xi)\exp[i(\kappa x - \omega t + \theta(\xi))]$. Then, the following conclusions hold.*

(i) For any $(\kappa, \omega) \in R^2$ in Figure 1 (a), corresponding to the families of the periodic orbits of system (1.8), equation (1.1) always has the exact explicit solutions with the parametric representations given by (4.1) or (4.2).

(ii) When $(\kappa, \omega) \in (II), (III), (IV)$ and $(L_2), (L_3)$ in Figure 1 (a), corresponding to the families of the periodic orbits of system (1.8), equation (1.1) has the exact explicit solutions with the parametric representations given by (4.3), (4.4), (4.5), (4.6) and (4.7).

(iii) When $(\kappa, \omega) \in (II), (III)$ and $(L_1), (L_2)$ in Figure 1 (a), corresponding to the heteroclinic orbits of system (1.8), equation (1.1) has the exact explicit solutions with the parametric representations given by (4.8), (4.9) and (4.10).