Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 5, Number 3, September 2023, 637–647 DOI:10.12150/jnma.2023.637

Asymptotic Expansion of Solutions to Singular Perturbation Problems in Critical Cases*

Hao Zhang[1] and Na Wang[1,†]

Abstract This paper investigates the problem of singular perturbed integral initial values and Robin boundary values in the critical case. Based on the boundary layer function method, we not only construct the asymptotic approximation of the original equation, but also prove the uniform validity of the asymptotic solution by successive approximation. At the same time, we give an example to prove the validity of the theoretical results.

Keywords Singularly perturbed problem, critical case, boundary function method, approximate solution

MSC(2010) 34B15, 34E10, 34E15.

1. Introduction

Singularly perturbed differential equation boundary value problems are frequently utilized as mathematical models to represent biomechanical and physical phenomena [1, 2, 18], such as chemical kinetics [19] and semiconductor simulation [20] phenomena. They correspond to mathematical models in which the degradation equation of singularly perturbed differential equation boundary value problems does not have an isolated root, but has a series of roots depending on one or more parameters instead. This case will be called the critical case [6]. Compared with the singularly perturbed initial boundary value problem with isolated roots of other degradation equations, the critical case has the following three difficulties: first, the zero order regular approximation solution is an unknown arbitrary function, which needs to be obtained by subsequent conditions; secondly, the solution process of the zero-order boundary layer is also very complicated; thirdly, the coupling of equations related to $k\,(k \geqslant 1)$-order boundary layer functions can be reduced by a specific diagonalization transformation. Therefore, it is very meaningful and valuable to study the critical case in singularly perturbed problems.

The research methods of singular perturbation critical problems mainly include boundary layer function method [13]. Through the theory of boundary layer function method, Vasil'eva and Butuzov [6] were the first to study initial value problems for singularly perturbed systems in the critical case. Subsequently, in recent years, some related issues in critical case have been solved [3–5, 7, 15, 17]. To the

†the corresponding author.
Email address: zhanghao717519@163.com(H. Zhang), wangna1621@126.com (N. Wang)
[1]College of Sciences, Shanghai Institute of Technology, Shanghai, 201418, China

*The authors were supported by Nature Science Fund of Shanghai Institute of Technology (NO: 1021GK210006141).

best of our knowledge, the above problems only pertain to Dirichlet or Neumann boundary value conditions, while the Robin boundary value conditions and integral initial value conditions [8] are seldom investigated. In the past several decades, authors [9–11,14,16] have discussed the singularly perturbed integral initial boundary value and Robin initial boundary value problems in various non-critical cases. However, until now, the singularly perturbed problem in critical cases with initial integral value and Robin boundary value condition has not been studied. Inspired by this, we fill in the gap of this kind of problem, and give formal asymptotic solutions and numerical examples.

The structure of this paper is as follows. The boundary layer function method [13] is used in sections 2 and 3 to introduce corresponding singular perturbation critical problems and to obtain the zero-order approximate solution and the first-order approximate solution of the original equation; Section 4 uses the successive approximation method [4,12] to demonstrate the existence of the solution; Section 5 provides an example to illustrate the second part of the theoretical results; the last section presents our conclusions.

2. Statement of the problem

Consider the following class of singularly perturbed initial boundary value problems

$$\begin{cases} \mu \frac{dz}{dt} = A\left(y,t\right)y + \mu B\left(z,t\right), \\ \mu \frac{dy}{dt} = C\left(t\right)y + \mu D\left(z,t\right), \end{cases} a \leqslant t \leqslant b, \tag{2.1}$$

the corresponding integral initial value and Robin boundary value condition

$$z(a) = z^0 + \int_a^b p_1\left(z\left(s,\mu\right)\right)ds, \qquad z(b) - z'\left(b\right) = z^1, \tag{2.2}$$

where z, y are scalar functions, and $0 < \mu \ll 1$ is a small parameter. Systems (2.1)-(2.2) must satisfy the following requirements:

(H_1) Assume that functions $A\left(y,t\right), B\left(z,t\right), C\left(t\right), D\left(z,t\right)$ and $p_1\left(z\right)$ are sufficiently smooth on $G = \{(z,y,t) \,||z| \leq l, |y| \leq l, a \leq t \leq b\}$, and l, z^0, z^1 are all real numbers. To be definite, assume that $C\left(t\right) < 0$.

The degradation equation of problem (2.1) is as follows.

$$\begin{cases} A\left(\bar{y},t\right)\bar{y} = 0, \\ C\left(t\right)\bar{y} = 0. \end{cases} \tag{2.3}$$

We can get from the degradation equation (2.3) that $\bar{y}\left(t\right) = 0$. However, since the degradation equation (2.3) does not include $\bar{z}\left(t\right)$, $\bar{z}\left(t\right)$ is an unknown function about the variable t. Assume that $\bar{z}\left(t\right) = \alpha\left(t\right)$ for the convenience of calculation. Then $\alpha\left(t\right)$ is also an arbitrary unknown function.

Set

$$F = \begin{pmatrix} A\left(y,t\right)y + \mu B\left(z,t\right) \\ C\left(t\right)y + \mu D\left(z,t\right) \end{pmatrix}, \qquad x = \left(z,y\right)^T.$$