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Improved Analysis of PINNs: Alleviate the CoD for Compositional Solutions

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Dedicated to the memory of Professor Zhongci Shi

Abstract. In this paper, we present an improved analysis of the Physics Informed Neural Networks (PINNs) method for solving second-order elliptic equations. By assuming an intrinsic sparse structure in the underlying solution, we provide a convergence rate analysis that can overcome the curse of dimensionality (CoD). Specifically, using some approximation theory in Sobolev space together with the multivariate Faa di Bruno formula, we first derive the approximation error for composition functions with a small degree of freedom in each compositional layer. Furthermore, by integrating several results on the statistical error of neural networks, we obtain a refined convergence rate analysis for PINNs in solving elliptic equations with compositional solutions. We also demonstrate the benefits of the intrinsic sparse structure with two simple numerical examples.

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1 Introduction

With recent success of neural networks in many areas of science computation [1–6], solving partial different equations(PDEs) with deep learning methods has been widely studied, e.g., the Physics-Informed Neural Networks (PINNs) [7], the Deep Ritz Method (DRM) [8] and the Weak Adversarial Neural Networks (WAN) [9]. While both DRM and WAN use the weak form of PDEs, PINNs solves PDEs by a direct minimization of the square residuals in strong form, which makes the method more flexible and easier to formulate for general problems. Unlike classical function approximation strategies like polynomial approximation or Fourier transforms, deep neural networks turns out to be an easy-to-implement tool which shows impressive expressive power in dealing with several high-dimensional problems, such as solving the Kolmogorov PDEs [10].

On the other hand, several theoretical works on the approximation theory of DNN and the convergence rage of PINNs have also been done. For instance, based on the localized polynomial approximation Yarotsky establishes an upper and lower bound for DNN with ReLU as an activation function [11]; Shen et al. study the approximation error for smooth functions [12] and Hölder continuous functions [13], resulting in a non-asymptotic estimation characterized by width and depth of the DNN. More recently, in the context of solving PDEs with deep learning method, the approximation theory of DNN has also been generalized to the Sobolev space. In [14] Gühring et al. extend Yarotsky's proof to the higher-order Sobolev space, with the help of approximate partition of unity and averaged Taylor polynomials. Hon et al. also obtain a simultaneous approximation result in Sobolev space for smooth functions [15], with a similar construction proof as in [12]. In the meanwhile, in [16, 17] the authors provide rigorous convergence analysis for PINNs and DRM by simultaneously controlling the approximation error and the generalization error of deep neural networks. According to these results, however, the curse of dimensionality (CoD) still exists in general situation if no additional assumption is made on the targeted function itself. In the past years, one such function space is the Barron classes. Inspired by the original work by Barron [18], E et al. studied the approximation ability of two-layer neural networks for the Barron space and residual neural network for the flow-induced function spaces [19], of which the convergence rate is independent of the dimensionality. Furthermore, Lu et al. [20] give an analysis of the DRM method for a two-layer neural network space, which also shows the CoD can be avoided if the solution is in the Barron space. More recently, they study some sufficient conditions for the equations to admit such a solution [21, 22].

In this work, we present an improved analysis for PINNs with compositional solutions. By assuming a sparse structure in each composition layer, we show that