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A Linearized Adaptive Dynamic Diffusion Finite Element Method for Convection-Diffusion-Reaction Equations

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Dedicated to the memory of Professor Zhongci Shi

Abstract. In this paper, we consider a modified nonlinear dynamic diffusion (DD) method for convection-diffusion-reaction equations. This method is free of stabilization parameters and capable of precluding spurious oscillations. We develop a reliable and efficient residual-type a posteriori error estimator, which is robust with respect to the diffusivity parameter. Furthermore, we propose a linearized adaptive DD algorithm based on the a posteriori estimator. Finally, we perform numerical experiments to verify the theoretical analysis and the performance of the adaptive algorithm.

AMS subject classifications: 65K10, 65N30, 65N21, 49M25, 49K20

Key words: Convection-diffusion-reaction equation, dynamical diffusion method, residual-type a posteriori error estimator, adaptive algorithm.

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1 Introduction

Let $\Omega\subset\mathbb{R}^d$ ($d=2,3$) be a polygonal/polyhedral domain with boundary $\Gamma=\bar{\Gamma}_D\cup\bar{\Gamma}_N$, where $\Gamma_D\cap\Gamma_N=\emptyset$ and $meas(\Gamma_D)>0$. We consider the convection-diffusion-reaction problem

$$\begin{cases} -\varepsilon\Delta u+\beta\cdot\nabla u+\sigma u=f & \text{in } \ \Omega, \\ u=0 & \text{on } \ \Gamma_D, \\ \varepsilon\nabla u\cdot\mathbf{n}=g & \text{on } \ \Gamma_N, \end{cases} \tag{1.1}$$

where u represents the quantity being transported, $0<\varepsilon\ll 1$ the (constant) diffusivity, $\beta\in[W^{1,\infty}(\Omega)]^d$ the velocity field, $0\leq\sigma\in L^\infty(\Omega)$ the reaction coefficient, $f\in L^2(\Omega)$ the source term, $\mathbf{n}$ the outward unit normal vector along Γ, and $g\in L^2(\Gamma_N)$ the Neumann boundary condition.

We first make the following two assumptions:

(D1) There are two nonnegative constants γ and c_σ, independent of ε, satisfying

$$\sigma-\frac{1}{2}\nabla\cdot\beta\geq\gamma \quad \text{and} \quad \|\sigma\|_{0,\infty}\leq c_\sigma\gamma, \tag{1.2}$$

this assumption ensures that we can simultaneously handle the cases of $\sigma\neq 0$ and $\sigma=0$ (cf. [53]), with the latter case corresponding to $\gamma=0$ and $c_\sigma=0$.

(D2) The Dirichlet boundary Γ_D includes the inflow boundary, i.e., $\Gamma_D\supset\{\mathbf{x}\in\Gamma: \beta(\mathbf{x})\cdot\mathbf{n}<0\}$.

It is well-known that classical finite element methods usually give rise to numerical oscillations for convection-dominated convection-diffusion equations, due to local singularities arising from interior or boundary layers (cf. [12–14, 23, 44, 45]). In order to overcome the limitations of the Galerkin mehtods, a lot of stabilized finite element methods have been developed, such as Streamline-Upwind/Petrov-Galerkin (SUPG) methods [9], Galerkin-Least-Squares methods [32], Residual-Free Bubbles methods [5], Variational Multiscale (VMS) methods [7,8,23,26–29,33,34,36,41], and Discontinuous Galerkin type methods [2, 10, 15, 18, 19, 24, 35]. By suitable designing stabilization parameters, these linear stabilized methods are capable of producing accurate and oscillation-free numerical solutions in regions where the exact solution has no abrupt changes. However, they do not preclude spurious oscillations in the neighbourhood of sharp layers.

To get rid of the local oscillations, many nonlinear stabilized finite element methods have been developed by adding an extra diffusivity term to the formulation to recover the monotonicity of the continuous problem [1, 11, 20, 25, 31, 37, 48–51]. Such