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A New Locking-Free Virtual Element Method for Linear Elasticity Problems

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Dedicated to the memory of Professor Zhongci Shi

Abstract. This paper devises a new lowest-order conforming virtual element method (VEM) for planar linear elasticity with the pure displacement/traction boundary condition. The main trick is to view a generic polygon K as a new one $\widetilde{K}$ with additional vertices consisting of interior points on edges of K, so that the discrete admissible space is taken as the V_1 type virtual element space related to the partition $\{\widetilde{K}\}$ instead of $\{K\}$. The method is proved to converge with optimal convergence order both in H^1 and L^2 norms and uniformly with respect to the Lamé constant λ. Numerical tests are presented to illustrate the good performance of the proposed VEM and confirm the theoretical results.

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1 Introduction

Robust numerical solution of parameter dependent partial differential equations (PDEs) is a very important topic in scientific computing. A typical example is

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the linear elasticity problem which involves the Lamé constants λ and μ. If λ is very large, the material becomes nearly incompressible, which will lead to the so-called volume locking phenomenon when the Courant element is used for discretization (cf. [8]). In the past four decades or more, a huge work has been developed to overcome this difficulty. We refer the reader to [6, 7, 15, 17, 19, 28, 29, 32, 38, 42] and the references therein for details along this line.

On the other hand, the virtual element method (VEM) is a newly proposed numerical method for solving PDEs in recent years, which can be viewed as a generalized finite element method allowing for the use of polytopal meshes (polygons or polyhedra) of very general shape. It was first proposed and analyzed in [2, 9, 11] and immediately attracted much attention from the research community due to its advantages in handling problems with complex geometries, high-regularity solutions (cf. [21, 31]). As far as we know, there have also developed some locking free VEMs for the linear elasticity problem. Let k denote the order of the virtual element used. The conforming VEMs for this problem were first studied in [10]. However, the condition $k\geq 2$ is required so as to produce locking free optimal error estimates in the H^1 norm. More recently, a lowest-order locking-free VEM was devised in [37] to remedy this disadvantage by modifying the W_1 type virtual element technically so that the uniform inf-sup condition holds. This method can be regarded as the extension of the well-known Bernardi-Raugel finite element method (cf. [14]) in general polygonal meshes. In [41], some nonconforming VEMs were proposed for attacking the previous problem in two and three dimensions. It was shown in this paper that if $k\geq 1$ (resp. $k\geq 2$), the method is locking-free for the pure displacement (resp. traction) problem. Recently, two kinds of lowest-order locking free VEMs were proposed in [33]. Here, the first one is nonconforming virtual element combined with a special stabilizing term, and the second one is constructed by using the conforming virtual element for one component of the displacement field and the nonconforming virtual element for the other. We mention in passing that there are some other VEMs discussing numerical solution of elasticity problems and other problems in different aspects; see, e.g., [3–5, 13, 18, 23, 25–27, 34, 35, 39, 40].

In this paper, we are concerned with proposing and analyzing a new lowest-order conforming locking-free VEM for the pure displacement/traction problem, by means of a significant feature of VEMs. The main ideas behind the method include:

1. For a generic polygonal element K with N_K edges, denoted by $\{z_i\}_{i=1}^{N_K}$ its vertices which are numbered in a counter-clockwise order and by $e_i=\overline{z_iz_{i+1}}$ the edge connecting z_i to z_{i+1}, where $z_{N_K+1}:=z_1$. Then we construct an auxiliary polygon $\widetilde{K}$ (see Fig. 1), which has the same geometric shape as K but has more vertices. Precisely speaking, besides the vertices of K, $\widetilde{K}$ has additional vertices formed by the internal points $\{m_i\}_{i=1}^{N_K}$ of edges of K, which