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On the Solution Accuracy Downstream of Shocks When Using Godunov-Type Schemes. I. Sources of Errors in One-Dimensional Problems

Alexander V. Rodionov*

Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Moscow, 125047, Russia.
RFNC – All-Russian Scientific Research Institute of Experimental Physics, Sarov, 607188, Russia.

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Abstract. The article opens a series of publications devoted to a systematic study of numerical errors behind the shock wave when using high-order Godunov-type schemes, including in combination with the artificial viscosity approach. The proposed paper describes the numerical methods used in the study, and identifies the main factors affecting the accuracy of the solution for the case of one-dimensional gas dynamic problems. The physical interpretation of the identified factors is given and their influence on the grid convergence is analyzed.

AMS subject classifications: 65M08, 65M12, 35L65, 35L67

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1 Introduction

When solving gas dynamic problems in the context of Euler equations, a shock wave is treated as a discontinuity, across which the Rankine-Hugoniot relations are valid. In shock-fitting techniques, shock waves (as well as contact discontinuities) are tracked explicitly, which in principle allows (in one-dimensional problems or in multidimensional problems with a simple wave configuration) obtaining a solution with a high accuracy. However, in multidimensional problems with complex flow structures the use of such methods is problematic.

*Corresponding author. *Email address:* `avrodionov@rambler.ru` (A. V. Rodionov)

In this connection, shock-capturing methods (schemes) that smear discontinuities over computing cells are in common use in computational fluid dynamics. In such a case, the shock front turns into a shock layer with a width of a few mesh spacing. Although the Rankine-Hugoniot conditions are not explicitly applied in such computations, one can expect that they are approximated integrally (over the shock layer) if the shock-capturing method approximates the conservation laws.

When a shock-capturing method is used, shock waves are smeared due to numerical viscosity (scheme dissipation), which in some sense mimics the effect of physical viscosity. Having functional similarity, numerical and physical viscosities have fundamental distinctions. So, taking into account the physical viscosity we approximate the Navier-Stokes equations within the shock layer. By refining the mesh, we improve the resolution of the shock layer, and the numerical solution tends to the exact solution of the Navier-Stokes equations. In the case of numerical viscosity, the mesh refinement does not lead to any gain in resolution of the shock layer, since its width decreases proportionally. Therefore, fundamental questions arise here: will the numerical solution with the mesh refinement converge to the exact solution of Euler equations supplemented by Rankine-Hugoniot relations across the shock, and, if so, what will be the rate of convergence?

A large number of publications are devoted to the study of such issues. Of the works known to the author, publications [1–16] are of particular interest. Among them, works [1, 4–6, 8, 9, 11, 13, 16] considered the Euler equations, while the remaining papers solved other nonlinear systems of hyperbolic equations, such as the shallow water equations.

Today, it is generally agreed that the convergence rate of high-order shock-capturing schemes reduces to the first order downstream of shocks. This assertion was verified numerically for the third-order Rusanov scheme [1], the second-order MUSCL-type schemes [3,5,9,16], the families of ENO and WENO schemes [2,4,5,9,11,12,14,16], the monotonicity preserving (MP) scheme [11, 16] and the discontinuous Galerkin (DG) method [15]. The relevant analytical studies can be found in [6–8,10,16]. To overcome this problem, the following techniques were suggested: the subcell resolution method [2], the matrix viscosity method [8], the fast sweeping approach [13] and the specific combined scheme [14]. However, these techniques were implemented only in one-dimensional model problems, and their generalization to complex multidimensional cases seems to be just as problematic as in the case of using shock-fitting techniques.

Although the problem of the solution accuracy behind the shock wave has been attracting the attention of researchers for several decades, it is not yet sufficiently studied. This is due to both the wide variety of shock-capturing methods and the problems solved with their help, and the complex nature of the error sources inherent in such methods.

The present article opens a series of publications in which the author intends to report the results of his systematic study of this problem in relation to high-order Godunov-type schemes, including in combination with the artificial viscosity approach [17–20] (this approach not only cures the carbuncle phenomenon, but also reduces substantially the post-shock oscillations). The proposed article (1) describes the numerical methods used in the study and (2), by the example of solving one-dimensional gas dynamic problems,