Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **15**, No. 6, pp. 1407-1427

DOI: 10.4208/aamm.OA-2021-0313
December 2023

A Quadratic Finite Volume Method for Parabolic Problems

Yuanyuan Zhang and Xiaoping Liu*

School of Mathematics and Information Sciences, Yantai University, Yantai, Shandong 264005, China

Received 5 October 2021; Accepted (in revised version) 10 March 2022

Abstract. In this paper, a quadratic finite volume method (FVM) for parabolic problems is studied. We first discretize the spatial variables using a quadratic FVM to obtain a semi-discrete scheme. We then employ the backward Euler method and the Crank-Nicolson method respectively to further disctetize the time vatiable so as to derive two full-discrete schemes. The existence and uniqueness of the semi-discrete and full-discrete FVM solutions are established and their optimal error estimates are derived. Finally, we give numerical examples to illustrate the theoretical results.

AMS subject classifications: 65N15, 65N30

Key words: Higher-order finite volume method, parabolic problems, error estimate.

1 Introduction

Lots of scientific and engineering processes can be described by parabolic equations, such as diffusion, biomechanics, environmental protection, etc. Finite element methods (FEMs) for solving parabolic problems have been deeply studied, see e.g., [1, 4, 7, 16, 18, 23, 24, 30, 32]. Compared with the FEM, the FVM has an obvious advantage of preserving local conservation laws, which is crucial for many physical and engineering applications. Due to its advantages, the FVM has become a popular numerical method for solving partial differential equations (PDEs), see e.g., [6, 19, 20, 22, 29]. The purpose of this paper is to study a quadratic FVM discretization method based on triangular meshes for solving parabolic problems.

The linear FVMs for solving PDEs have been studied a lot and many results have been derived, see e.g., [3, 10, 17, 21]. Even though higher-order FVMs have great challenges in theoretical analysis compared with linear FVMs, they can obtain higher order convergence accuracy and have attracted many scholars' attention. The research on

*Corresponding author.
Emails: yyzhang@ytu.edu.cn (Y. Zhang), xpliu123@126.com (X. Liu)

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higher-order FVMs for solving elliptic problems has made great progress in recent years, see e.g., [5, 9, 25, 26, 31, 33]. For parabolic problems, Gao and Wang in [13] established the super-convergence property of a cubic FVM for one-dimensional parabolic equations. Yu and Li in [28] used optimal stress points to develop a biquadratic FVM on quadrilateral meshes. Yang, Liu and Zou in [27] presented a unified analysis of high order FVMs on quadrilateral meshes and derived their optimal error estimates. To our best knowledge, there is little work about higher-order FVMs based on triangular meshes for solving parabolic problems.

Most of the existing higher-order FVMs based on triangular meshes require that the primary meshes satisfy certain minimal angle conditions to ensure its optimal error estimates, see e.g., [8, 26]. However, Zou in [33] first proposed a quadratic FV scheme which possesses the optimal H^1-norm error estimate over any shape regular triangular mesh without any additional minimal angle conditions. What's more, under a novel mapping from the trial space to the test space, its bilinear form can be regarded as a small perturbation of the corresponding quadratic FEM. This fact might greatly simplify its theoretical analysis.

In this paper, we discretize the spacial variables of the parabolic problems adopting the quadratic FVM developed in [33] for elliptic problems. The FVMs for elliptic problems come down to systems of linear equations, so that the inf-sup condition can guarantee the existence and uniqueness of their solutions and optimal error estimates, whereas the semi-discrete FVMs for parabolic equations are converted into ordinary differential equations. Hence, in order to obtain the existence and uniqueness of the solutions of the semi-discrete FVM, we need to prove that the mass matrix is nonsingular. In addition, to derive error estimates of the semi-discrete FVM, we introduce an elliptic projection operator. Then, the error can be written as the sum of two terms. One of them is the error between the exact solution and its projection (denoted by ρ) and the other is the error between the projection and the solution of the FVM (denoted by e). The error ρ can be easily estimated using the results presented in [33]. However, the error e is much more difficulty and we spent a lot of effort to deal with it. Fortunately, we get that its solution can reach optimal error estimate over any shape regular triangular mesh. We further employ the backward Euler method and the Crank-Nicolson method to discretize the time variables to get two full-discrete FVMs. Similar to the error estimate of semi-discrete solution, we mainly focus on estimating the term e and derive that the convergence order of the backward Euler full-discrete FVM reaches 2 in space variable and 1 in time variable, while the Crank-Nicolson full-discrete FVM enjoys the optimal convergence order of 2 in both space and time variables.

The rest of the paper is organized as follows. In Section 2, we present a quadratic FVM for solving parabolic problems and give the semi-discrete scheme. Section 3 is devoted to the theoretical analysis of the semi-discrete FVM, including the existence and uniqueness of the solution and the error estimate. In Section 4, we introduce the backward Euler full-discrete scheme and the Crank-Nicolson full-discrete scheme and give their error estimates. Finally, we present some numerical experiments in Section 5.