JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **36**, No. 4, pp. 349-364

doi: 10.4208/jpde.v36.n4.2
December 2023

Asymptotic Stability for a Quasilinear Viscoelastic Equation with Nonlinear Damping and Memory

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Received 9 October 2020; Accepted 11 May 2023

Abstract. This paper is concerned with the asymptotic behavior of a quasilinear viscoelastic equation with nonlinear damping and memory. Assuming that the kernel $\mu(s)$ satisfies

$$\mu'(s) \leqslant -k_1\mu^m(s), \quad 1\leqslant m<\frac{3}{2},$$

we establish the exponential stability result for $m=1$ and the polynomial stability result for $1<m<\frac{3}{2}$.

AMS Subject Classifications: 35B40, 35L70

Chinese Library Classifications: O175.27

Key Words: Exponential stability; polynomial stability; quasilinear; nonlinear damping; memory.

1 Introduction

In this paper, we investigate the long-time dynamics of solutions for the quasilinear viscoelastic equation with nonlinear damping and memory

$$|u_t|^\rho u_{tt}-\Delta u_{tt}-\alpha\Delta u+\int_{-\infty}^{t}\mu(t-s)\Delta u(s)\mathrm{d}s+f(u)+g(u_t)=0 \tag{1.1}$$

in the unknown $u=u(x,t):\Omega\times\mathbb{R}^+\to\mathbb{R}$, complemented with the Dirichlet boundary condition

$$u(x,t)=0, \qquad (x,t)\in\partial\Omega\times\mathbb{R}^+, \tag{1.2}$$

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and the initial conditions

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \qquad x\in\Omega, \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^N(N\geqslant 1)$ with smooth boundary $\partial\Omega$, u_0 is the prescribed past history of u.

Eq. (1.1) provides a generalization, accounting for memory effects in the material, of equations of the form

$$f(u_t)u_{tt}-\Delta u-\Delta u_{tt}=0. \tag{1.4}$$

When $f(u_t)$ is a constant, Eq. (1.4) was introduced to model extensional vibrations of thin rods [1, Chapter 20] and ion-sound waves [2, Section 6]. There have been extensive researches on the well-posedness and the longtime dynamics for Eq. (1.4) with a different kind of damping term and source term, (see [3,4] and the references therein). When $f(u_t)$ is not a constant, Eq. (1.4) can model materials whose density depends on the velocity u_t [5]. We refer the reader to Fabrizio and Morro [6] for several other related models.

Let us recall some results concerning quasilinear viscoelastic wave equations with finite memory. In [7], the authors studied the following equation with Dirichlet boundary conditions

$$|u_t|^\rho u_{tt}-\Delta u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(s)\mathrm{d}s-\gamma\Delta u_t=0. \tag{1.5}$$

By assuming

$$0<\rho\leqslant\frac{2}{N-2} \quad \text{if } N\geqslant 3, \qquad \rho>0 \quad \text{if } N=1,2, \tag{1.6}$$

they proved a global existence result for $\gamma\geqslant 0$ and an exponential decay result for $\gamma>0$. In the absence of the strong damping $(\gamma=0)$, Messaoudi and Tatar [8] established the exponential and polynomial decay rates of energy. Messaoudi and Mustafa [9] improved the results in [8] and proved an explicit and general energy decay formula that allows a larger class of functions $g(s)$. Recently, based on integral inequalities and multiplier techniques, Li and Hu [10] proved a general decay rate from which the exponential decay and the polynomial decay are only special cases.

In [11], Messaoudi and Tatar studied the following problem

$$|u_t|^\rho u_{tt}-\Delta u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(s)\mathrm{d}s=b|u|^{p-2}u, \tag{1.7}$$

with boundary and initial conditions (1.2) and (1.3). By introducing a new functional and using a potential well method, they obtained the global existence of solutions and the uniform decay of the energy if the initial data are in some stable set. When the only dissipation effect is given by the memory (i.e. $b=0$ in (1.7)), Messaoudi and Tatar [12] proved the exponential decay of global solutions to (1.7), without smallness of initial data. Liu [13] considered a system of two coupled quasilinear viscoelastic equations in canonical form with Dirichlet boundary condition. Using the perturbed energy method,