INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 6, Pages 772–804

doi: 10.4208/ijnam2023-1034

A POSTERIORI ERROR ESTIMATES FOR A LOCAL DISCONTINUOUS GALERKIN APPROXIMATION OF SEMILINEAR SECOND-ORDER ELLIPTIC PROBLEMS ON CARTESIAN GRIDS

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Abstract. In this paper, we design and analyze new residual-type *a posteriori* error estimators for the local discontinuous Galerkin (LDG) method applied to semilinear second-order elliptic problems in two dimensions of the type $-\Delta u = f(\mathbf{x}, u)$. We use our recent superconvergence results derived in *Commun. Appl. Math. Comput. (2021)* to prove that the LDG solution is superconvergent with an order $p+2$ towards the p-degree right Radau interpolating polynomial of the exact solution, when tensor product polynomials of degree at most p are considered as basis for the LDG method. Moreover, we show that the global discretization error can be decomposed into the sum of two errors. The first error can be expressed as a linear combination of two $(p+1)$-degree Radau polynomials in the x- and $y-$ directions. The second error converges to zero with order $p+2$ in the L^2-norm. This new result allows us to construct *a posteriori* error estimators of residual type. We prove that the proposed *a posteriori* error estimators converge to the true errors in the L^2-norm under mesh refinement at the optimal rate. The order of convergence is proved to be $p+2$. We further prove that our *a posteriori* error estimates yield upper and lower bounds for the actual error. Finally, a series of numerical examples are presented to validate the theoretical results and numerically demonstrate the convergence of the proposed *a posteriori* error estimators.

Key words. local discontinuous Galerkin method, semilinear elliptic problems, *a posteriori* error estimators, superconvergence, Radau polynomial.

1. Introduction

The *a posteriori* error estimates are computable quantities from numerical solutions. They can be used for mesh modification such as refinement or coarsening [50]. In this work, we design and analyze *a posteriori* error estimators for the local discontinuous Galerkin (LDG) for the following semilinear second-order elliptic problems of the form

$$-\Delta u = f(\mathbf{x}, u), \quad \mathbf{x} \in \Omega. \tag{1a}$$

In our analysis we assume that the nonlinear function $f: \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is smooth with respect to its arguments. Since *a priori* error estimates provided in [25] will be used, we make the same assumption on f. To be more precise, we always assume that f and its partial derivatives are continuous for $\mathbf{x} \in \overline{\Omega}$ and $u \in \mathbb{R}$ and satisfies the following uniform bound

$$|f(\mathbf{x}, u)| \le M, \quad \forall\, \mathbf{x} \in \Omega, \quad \forall\, u \in \mathbb{R}, \tag{1b}$$

as well as the Lipschitz condition

$$|f_u(\mathbf{x}, u) - f_u(\mathbf{y}, v)| \le L\left(\|\mathbf{x}-\mathbf{y}\| + |u-v|\right), \quad \forall\, \mathbf{x},\ \mathbf{y} \in \Omega, \quad \forall\, u,\ v \in \mathbb{R}. \tag{1c}$$

We focus on two dimensions and write $\mathbf{x}$ as (x, y). Without loss of generality, we consider a rectangular domain denoted by $\Omega = \{\mathbf{x} = (x, y) :\ a < x < b,\ c < y < d\}$.

Received by the editors on June 19, 2022 and, accepted on July 4, 2023.
2000 *Mathematics Subject Classification.* 65N12, 65N15, 65N30, 65N50.

Here, we remark that our results remain true, with minor changes in the proofs, when Ω is a rectangular bounded domain of $\mathbb{R}^3$. In this paper, we will consider either periodic boundary conditions

$$u(a,y) = u(b,y),\ u(x,c) = u(x,d),$$
$$u_x(a,y) = u_x(b,y), \quad u_y(x,c) = u_y(x,d),\ (x,y) \in \partial\Omega, \tag{1d}$$

or purely Dirichlet boundary conditions

$$u = g_D, \quad (x,y) \in \partial\Omega, \tag{1e}$$

or mixed Dirichlet-Neumann boundary conditions

$$u = g_D, \quad (x,y) \in \partial\Omega_D, \quad \mathbf{n} \cdot \nabla u = \mathbf{n} \cdot \mathbf{g}_N, \quad (x,y) \in \partial\Omega_N. \tag{1f}$$

Here, $\mathbf{n}$ is the outward unit normal to the boundary, $\partial\Omega$, of Ω. For the mixed boundary conditions (1f), we make the assumption that the boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ is decomposed into two disjoint sets denoted by $\partial\Omega_D$ and $\partial\Omega_N$, where Dirichlet and Neumann boundary conditions are imposed, respectively. In addition, we assume that the measure of the Dirichlet boundary $\partial\Omega_D$ is nonzero. In our analysis, we assume that the given functions f, g_D, and $\mathbf{g}_N$ are smooth functions on their domains such that the problem (1) has one and only one solution $u \in H^2(\Omega)$. We refer the reader to [37, 39, 41] and the references therein for the existence and uniqueness of solutions to general elliptic problems.

Among the numerous numerical schemes used to solve elliptic problems, the discontinuous Galerkin (DG) finite element methods constitute an important class. In recent years, DG methods have been proven to be powerful and popular computational methods for the numerical solution of partial differential equations. They have been successfully applied to approximate solutions to many linear and nonlinear time-independent as well as time-dependent problems. The DG method was originally proposed by Reed and Hill in [46] to solve hyperbolic conservation laws with only first-order spatial derivatives. A major development of the DG method is the so-called Runge-Kutta DG (RKDG) framework proposed for solving nonlinear hyperbolic conservation laws containing first order spatial derivatives in a series of papers by Cockburn, Shu *et al.*. Later, serval DG methods were designed to deal with equations involving higher order derivatives. The DG method has many attractive features compared with the classical numerical methods such as finite difference and finite element methods. The main advantages of these DG methods include the high order accuracy, geometric flexibility, suitability for $h-$ and $p-$adaptivity, extremely local data structure, high parallel efficiency and a good theoretical foundation for stability and error estimates. We refer the reader to [33, 45, 47, 48] for more information on many DG methods and their applications.

The DG method was later generalized to the so-called local discontinuous Galerkin (LDG) method by Cockburn and Shu to solve problems with higher order spatial derivatives, such as convection-diffusion equations [35, 51], wave equations [10, 13], and other third- and fourth-order problems [36, 44, 53]. The LDG method shares all the nice features of the DG methods for hyperbolic equations, and it becomes one of the most popular numerical methods for solving elliptic problems. The main idea of the LDG method is to convert the original differential equation into a system of first-order differential equations by introducing some auxiliary variables, and then discretize the resulting system with the classical DG method for first-order equations. With carefully chosen numerical fluxes, the stability and convergence of the LDG methods have been studied for many linear and nonlinear model problems. After that, the LDG method has been a popular way to solve many problems with