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Global Bifurcation for a Class of Lotka-Volterra Competitive Systems*

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Abstract For a class of Lotka-Volterra competitive systems including both diffusion and advection, a global bifurcation result of positive steady states is established via a bifurcation approach. Also, the phenomenon of multiple positive steady states is discussed.

Keywords Competitive system, bifurcation, positive steady state, multiple solutions

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1. Introduction

Over the past several decades, the Crandall-Rabinowitz local bifurcation theorem [4] and the global bifurcation theorem [24] have been widely utilized to understand the solution set of nonlinear equations and to reveal critical roles played by physical or biological parameters (see, e.g., [5,17] for a class of nonlocal elliptic equations, [1,13,19,33,34] for the two-species reaction-diffusion competition models and [18,28,31] for the predator-prey-taxis models). For more investigations, we refer the interested readers to [16,20,21,29,30,35,39] (to mention but a few).

In this paper, we are mainly interested in the following general competitive parabolic system including both diffusion and advection

$$\begin{cases} u_t = \mathcal{L}_1 u + u[r_1(x) - u - bv], & x \in \Omega, \ t > 0, \\ v_t = \mathcal{L}_2 v + v[r_2(x) - cu - v], & x \in \Omega, \ t > 0, \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x) \geq, \not\equiv 0, & x \in \Omega, \\ v(x,0) = v_0(x) \geq, \not\equiv 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^N$ with $1 \leqslant N \in \mathbb{Z}$, $u(x,t)$ and $v(x,t)$ represent the population density of two competing species at location $x \in \Omega$

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and time $t > 0$, respectively, $r_i(x)$ $(i = 1, 2)$ is a bounded and positive function accounting for the intrinsic growth rate, and the positive constants b and c measure the inter-specific competition intensities (note that the intra-specific competition coefficients have been normalized by 1). The linear differential operator $\mathcal{L}_i$ is defined by

$$\mathcal{L}_i w := \operatorname{div}\Big(d_i \nabla w - \alpha_i w \nabla P_i(x)\Big), \quad i = 1, 2, \tag{1.2}$$

with $d_i > 0$ denoting the rate of random diffusion, $P_i(x) \in C^1(\overline{\Omega})$ specifying the advective direction and $\alpha_i \in \mathbb{R}$ measuring the advection speed. The boundary operator $\mathcal{B}_i$ is defined by

$$\mathcal{B}_i w = d_i \frac{\partial w}{\partial \nu} - \alpha_i w \frac{\partial P_i}{\partial \nu} = 0, \quad i = 1, 2, \tag{1.3}$$

where ν denotes the outward unit normal vector on the boundary $\partial\Omega$. The no-flux boundary conditions imposed in (1.3) mean that no individuals can cross over the boundary of the habitat, i.e., the environment is closed.

Recently, system (1.1)-(1.3) has been systematically investigated by Zhou et al., [37,38], where the competition coefficients b and c are chosen as bifurcation/variable parameters, and the global dynamics is determined in a certain range of b and c on the b-c plane. Specifically, the authors first gave a clear picture of the local stability around the two semi-trivial steady states in terms of critical competition coefficients by the principal eigenvalue theory, then established an important estimate on the linear stability of any positive steady state via an analytic argument (see also a similar result by Guo, He and Ni [9]), and finally obtained the global dynamics in a certain range of b and c by appealing to the theory of monotone dynamical systems [10,11,14]. The main result suggests that either one of the two semi-trivial steady state is globally asymptotically stable (competitive exclusion) or there is a unique positive steady state which is globally asymptotically stable (coexistence), depending on the competition intensities (see details in [37, Theorems 4 and 5]).

To some extent, the works [37, 38] can be seen as a study on the parameter region of b and c where the global dynamics of system (1.1)-(1.3) can be completely determined. In the current paper, as a further development of [37, 38], we pursue further to understand the complicated dynamics of system (1.1)-(1.3), especially the structure of positive steady states. To this end, we primarily employ the bifurcation approach to present a global result on the structure of positive steady states.

Here, we mention several bifurcation results by considering some special cases or variants of system (1.1)-(1.3). For example, system (1.1)-(1.3) with $d_1 = d_2 = 1$, $\alpha_1 = \alpha_2 = 0$, $r_1 = r_2 = a > 0$ and $b, c > 1$ (the strong competition case) and with zero Dirichlet boundary conditions (as well as Neumann and Robin boundary conditions) have been investigated by Gui and Lou [8], where the existence and multiplicity of positive steady states are carefully examined by both bifurcation approach and degree method. A spatially one-dimensional case of system (1.1)-(1.3) together with Danckwerts boundary conditions, modeling the competition between two aquatic species in a free-flow environment, was studied by Wang, Tian, and Nie [32], where among other things, a picture on the structure of positive steady states is given by bifurcation approach. Moreover, Cantrell et al., [3] investigated