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Nonconforming Finite Elements for the $-\mathrm{curl}\Delta\mathrm{curl}$ and Brinkman Problems on Cubical Meshes

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Abstract. We propose two families of nonconforming elements on cubical meshes: one for the $-\mathrm{curl}\Delta\mathrm{curl}$ problem and the other for the Brinkman problem. The element for the $-\mathrm{curl}\Delta\mathrm{curl}$ problem is the first nonconforming element on cubical meshes. The element for the Brinkman problem can yield a uniformly stable finite element method with respect to the viscosity coefficient ν. The lowest-order elements for the $-\mathrm{curl}\Delta\mathrm{curl}$ and the Brinkman problems have 48 and 30 DOFs on each cube, respectively. The two families of elements are subspaces of $H(\mathrm{curl};\Omega)$ and $H(\mathrm{div};\Omega)$, and they, as nonconforming approximation to $H(\mathrm{grad}\,\mathrm{curl};\Omega)$ and $[H^1(\Omega)]^3$, can form a discrete Stokes complex together with the serendipity finite element space and the piecewise polynomial space.

AMS subject classifications: 65N30, 35Q60, 65N15, 35B45

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1 Introduction

Let $\Omega\subset\mathbb{R}^3$ be a contractible Lipschitz polyhedral domain. For $\boldsymbol{f}\in H(\mathrm{div}^0;\Omega)$, we consider the following $-\mathrm{curl}\Delta\mathrm{curl}$ problem:

$$\begin{aligned} -\mu\,\mathrm{curl}\Delta\mathrm{curl}\boldsymbol{u}+\mathrm{curl}\,\mathrm{curl}\boldsymbol{u}+\gamma\boldsymbol{u}&=\boldsymbol{f} && \text{in } \Omega,\\ \mathrm{div}\boldsymbol{u}&=0 && \text{in } \Omega,\\ \boldsymbol{u}\times\boldsymbol{n}&=0 && \text{on } \partial\Omega,\\ \mathrm{curl}\boldsymbol{u}&=0 && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

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Here $\gamma \geq 0$ is a constant of moderate size, $\mu > 0$ is a constant that can approach 0, $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega$, and $H(\mathrm{div}^0;\Omega)$ is the space of $[L^2(\Omega)]^3$ functions with vanishing divergence, i.e.,

$$H(\mathrm{div}^0;\Omega) := \{\boldsymbol{u} \in [L^2(\Omega)]^3 : \mathrm{div}\,\boldsymbol{u} = 0\}.$$

Problem (1.1) arises in applications related to electromagnetism and continuum mechanics [6,27,32]. Conforming finite element approximations of this problem require the construction of finite element spaces that belong to $H(\mathrm{grad\,curl};\Omega)$, which are commonly referred to as $H(\mathrm{grad\,curl})$-conforming finite element spaces. Recently, two of the authors, along with their collaborators, developed three families of $H(\mathrm{grad\,curl})$-conforming elements on both triangular and rectangular meshes [21,22,41]. The corresponding spectral construction of the three families of rectangular elements is detailed in [36]. In three-dimensional space, two of the authors proposed a tetrahedral $H(\mathrm{grad\,curl})$-conforming element [42] consisting of 315 DOFs per element, which was later improved by enriching the shape function space with piecewise-polynomial bubbles to reduce the DOFs to 18 [23]. See also [7,8] for a systematical construction of $H(\mathrm{grad\,curl})$-conforming elements on simplicial meshes. While the construction of $H(\mathrm{grad\,curl})$-conforming elements in two dimensions and on tetrahedral meshes is relatively complete, the development of cubical elements remains a challenge. The only cubical $H(\mathrm{grad\,curl})$-conforming element in the literature has 144 DOFs [35]. To address the issue of high DOFs in cubical elements, nonconforming finite elements may offer a viable solution. The existing literature reports two low-order nonconforming elements [24,48] and two $H(\mathrm{curl})$-conforming but $H(\mathrm{grad\,curl})$-nonconforming elements [25,40] on tetrahedral meshes. However, as far as we are aware, there has been no previous research on the construction of nonconforming cubical elements, which is one objective of this paper.

A related problem is the Brinkman model of porous flow, which seeks $(\boldsymbol{u};p)$ such that

$$\begin{aligned} -\mathrm{div}(\nu\,\mathrm{grad}\,\boldsymbol{u}) + \alpha\boldsymbol{u} + \mathrm{grad}\,p &= \boldsymbol{f} && \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{u} &= g && \text{in } \Omega, \\ \boldsymbol{u} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

Here $\boldsymbol{u}$ is the velocity, p is the pressure, $\alpha > 0$ is the dynamic viscosity divided by the permeability, $\nu > 0$ is the effective viscosity, and $\boldsymbol{f} \in [L^2(\Omega)]^3$ and $g \in L^2(\Omega)$ are two forcing terms. We assume α is a moderate constant, ν is a constant that can approach 0, and g satisfies the compatibility criterion $\int_\Omega g\,\mathrm{d}V = 0$. The Brinkman problem is used to describe the flow of viscous fluids in porous media with fractures. Applications of this model include the petroleum industry, the automotive industry, underground water hydrology, and heat pipes modeling. Depending on the value of effective viscosity ν, the Brinkman problem can be locally viewed as a Darcy or Stokes problem. When the Brinkman problem is Darcy-dominating (ν tends to 0), applying stable Stokes finite element pairs such as the Crouzeix-Raviart element [11], the Mini element [3], and the Taylor-Hood elements [34] will lead to non-convergent discretizations. Similarly, when