J. Math. Study
doi: 10.4208/jms.v56n4.23.03

Vol. **56**, No. 4, pp. 345-356
December 2023

Two-Weighted Estimate for Generalized Fractional Integral and its Commutator on Generalized Fractional Morrey Spaces

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Received August 11, 2022; Accepted Novermber 21, 2022
Published online December 13, 2023.

Abstract. The aim of this paper is to establish the mapping properties of generalized fractional integral I_ρ and its commutator $[b,I_\rho]$ formed by $b\in \mathrm{BMO}(\mathbb{R}^n)$ and the I_ρ on generalized fractional weighted Morrey spaces $\mathcal{L}_\omega^{p,\eta,\varphi}(\mathbb{R}^n)$, where φ is a positive and non-decreasing function defined on $(0,\infty)$, $\eta\in(0,n)$ and $p\in[1,\frac{n}{\eta})$.

AMS subject classifications: 42B20, 42B35, 47B47

Key words: Generalized fractional integral, commutator, space BMO($\mathbb{R}^n$), two-weight, generalized fractional Morrey space.

1 Introduction

Let I_α be the fractional integral operator of order $\alpha\in(0,n)$, i.e., for all $x\in\mathbb{R}^n$, define

$$I_\alpha f(x):=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\mathrm{d}y,$$

which, regards as the Hardy-Littlewood-Sobolev theorem (see [8, 23]), is bounded from spaces $L^p(\mathbb{R}^n)$ into spaces $L^q(\mathbb{R}^n)$, where $1<p<\frac{n}{\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Since then, the boundedness of I_α on various of spaces is widely focused. For example, Adams [1] has proved that I_α is bounded on Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey (see [17]). In 1994, Nakai has obtained the boundedness of I_α on generalized Morrey spaces $L^{p,\omega}(\mathbb{R}^n)$ in [18], where $\omega(a,r)=\omega(I)=\int_{I(a,r)}\omega(x)\mathrm{d}x$ and $I=I(a,r)$ is a cube $\{x\in\mathbb{R}^n:|x_i-a_i|\le\frac{r}{2},i=1,\cdots,n\}$. The further development about the operator I_α, the readers can see [4,12,15,16,26,29] and the corresponding references.

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To solve the $L^2(\mathbb{R}^n)$-boundedness of the Cauchy type integral along the Lipschitz curve, in 1965, Calderón [2] first introduced the following commutator

$$[b,T](f)=bT(f)-T(bf),$$

which is called Calderón commutator; furthermore, the boundedness of commutator $[b,H]$ generated by Hilbert transform H and functions $b\in \mathrm{BMO}(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$ is obtained. In 1976, Coifman-Rochberg-Weiss [6] have proved that the commutator $[b,T]$ generated by $b\in\mathrm{BMO}(\mathbb{R}^n)$ and a Calderón-Zygmund operator T is bounded on spaces $L^p(\mathbb{R}^n)$ for $1<p<\infty$. In 1982, Chanillo [3] has proved that the commutator $[b,I_\alpha]$ generated by $b\in\mathrm{BMO}(\mathbb{R}^n)$ and the I_α is bounded on from Lebesgue spaces $L^p(\mathbb{R}^n)$ into spaces $L^q(\mathbb{R}^n)$, where $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ with $1<p<\frac{n}{\alpha}$ and $0<\alpha<n$. Recently, the bounded properties of commutator $[b,I_\alpha]$ on different function spaces have been widely studied; for example, in 2022, Jia *et al.* [10] study the boundedness of fractional integrals I_α on special John-Nirenberg-Campanato spaces. In 2021, Tao *et al.* [25] have showed that the commutator $[b,T_\Omega]$ is compact on a ball Banach function space on $X\subset\mathbb{R}^n$ if and only if $b\in\mathrm{CMO}(\mathbb{R}^n)$. The other researches on the $[b,I_\alpha]$ can be seen [5,20,22,28] and their references therein.

In this paper, we will mainly study the boundedness of the generalized fractional integral operator I_ρ introduced in [19] and its commutator $[b,I_\rho]$ on generalized fractional weighted Morrey spaces $\mathcal{L}^{p,\eta,\varphi}_{\omega}(\mathbb{R}^n)$. Under assumption that the ρ and two-weight (ω,ν) satisfy certain conditions, in 2022, Ho [9] shows that potential type operators are bounded on weighted Morrey spaces, on the basis of this, we will show that the generalized fractional integral operator I_ρ is bounded on from generalized fractional weighted Morrey spaces $\mathcal{L}^{p,\eta,\varphi}_{\nu^p}(\mathbb{R}^n)$ into spaces $\mathcal{L}^{p,\eta,\varphi}_{\omega^p}(\mathbb{R}^n)$; furthermore, the boundedness of the commutator $[b,I_\rho]$ generated by $b\in\mathrm{BMO}(\mathbb{R}^n)$ and I_ρ on generalized fractional weighted Morrey spaces is also obtained.

Before stating the main results of this paper, we need to recall some necessary notations. The following definition of Muckenhoupt weight functions A_p is from [7].

Definition 1.1. *Let $p\in(1,\infty)$. A non-negative function $\omega\in L^1_{\mathrm{loc}}$ is said to be in the Muckenhoupt class A_p if there exists some positive constant C such that, for all balls $B\subset\mathbb{R}^n$,*

$$\left(\frac{1}{|B|}\int_B\omega(x)\mathrm{d}x\right)^{\frac{1}{p}}\left\{\frac{1}{|B|}\int_B[\omega(x)]^{1-p'}\mathrm{d}x\right\}^{\frac{1}{p'}}\le C. \tag{1.1}$$

And a weight ω is called an A_1 weight if there exists some positive constant C such that, for all balls $B\subset\mathbb{R}^n$,

$$\frac{1}{|B|}\int_B\omega(x)\mathrm{d}x\le C\inf_{y\in B}\omega(y).$$

Moreover, we define $A_\infty:=\bigcup_{p=1}^{\infty}A_p$.

Now we recall the definition of space BMO introduced in [11] as follows.