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A Multigrid Discretization of Discontinuous Galerkin Method for the Stokes Eigenvalue Problem

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Abstract. In this paper, based on the velocity-pressure formulation of the Stokes eigenvalue problem in d-dimensional case $(d=2,3)$, we propose a multigrid discretization of discontinuous Galerkin method using $\mathbb{P}_k-\mathbb{P}_{k-1}$ element $(k\geq 1)$ and prove its a priori error estimate. We also give the a posteriori error estimators for approximate eigenpairs, prove their reliability and efficiency for eigenfunctions, and also analyze their reliability for eigenvalues. We implement adaptive calculation, and the numerical results confirm our theoretical predictions and show that our method is efficient and can achieve the optimal convergence order $\mathcal{O}(dof^{\frac{-2k}{d}})$.

AMS subject classifications: 65N25, 65N30

Key words: Stokes eigenvalue problem, discontinuous Galerkin method, multigrid discretizations, adaptive algorithm.

1 Introduction

The Stokes eigenvalue problem is of great importance because of its role in stability analysis in fluid mechanics. For the Stokes eigenvalue problem, the development of numerical methods, as a model for incompressible fluid flow, is of great interest. B. Mercier et al. [34] first discussed the Stokes eigenvalue approximation by mixed method using the velocity-pressure formulation. Since then, more and more researchers entered this field and obtained many good results. In order to ensure the stability of numerical methods, the discrete inf-sup condition is necessary. However, to fit the discrete inf-sup condition, the discrete velocity and pressure spaces are dependent on each other. Hence,

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most papers on conforming finite element and non-conforming finite element methods appeared based on the velocity-pressure formulation using low-order elements, for example, see [3, 21, 26, 30, 31, 41, 43, 47]. Among them, [3, 21, 30, 31, 43, 47] adopted low-order conforming finite element methods to study the Stokes eigenvalue problem, [26,41] discussed the Stokes eigenvalue problem by using low-order non-conforming finite elements.

In order to use high-order elements, scholars adopted other formulations to study the Stokes eigenvalue problem, for instance, *Ö*nder T*ü*rk et al. [43] introduced a stable term by using the stress-displacement-pressure formulation, Meddahi et al. [33] proposed a finite element analysis of a pseudostress formulation, Gedicke et al. [16] conducted the a posteriori error analysis for the Arnold-Winther mixed finite element method using the stress-velocity formulation, and Lepe et al. [27] researched a virtual element approximation for the pseudostress formulation.

The discontinuous Galerkin finite element method (abbreviated as DGFEM) is one of the main numerical methods for solving partial differential equations (see [2,8,9,13,15,25, 36,38,45] and so on). For the Stokes eigenvalue problem, Gedicke et al. [17] discussed the divergence-conforming DGFEM using the Raviart-Thomas element based on velocity-pressure formulation on shape-regular rectangular meshes, Sun et al. [40, 42] discussed the mixed DGFEM using $\mathbb{P}_k-\mathbb{P}_{k-1}$ element based on velocity-pressure formulation, and Lepe et al. [28] discussed symmetric and nonsymmetric discontinuous Galerkin methods for a pseudostress formulation.

In this paper, we explore a multigrid discretization of the DGFEM using $\mathbb{P}_k-\mathbb{P}_{k-1}$ element for any $k\geq 1$ based on the velocity-pressure formulation of the Stokes eigenvalue problem. Our work has the following features:

(1) The two-grid discretization is an efficient method that was first introduced by Xu [48, 49] for nonsymmetric and nonlinear elliptic problems. In 1999, Xu and Zhou [50] applied this method to eigenvalue problems, and after that many scholars developed multigrid discretization method for eigenvalue problems (see, e.g., [11,12,19,21,23,29,41, 47,51]). In this paper, we propose a multigrid discretization scheme based on the shifted-inverse iteration for the DGFEM of the Stokes eigenvalue problem. With the scheme, the solution of the Stokes eigenvalue problem is reduced to the solution of an eigenvalue problem on a coarse mesh π_{h_0} and the solution of a series of linear algebraic systems on the fine meshes π_{h_l}. Theorems 3.1 and 3.2 in this paper ensure that the resulting solution can maintain an asymptotically optimal accuracy. The difficulty of analysis is $\mathcal{A}_h(\cdot,\cdot)$ is an inner product in the discontinuous finite element space, but not in the broken Sobolev space.

(2) In practical finite element computations, it is desirable to carry out the finite element computations in an adaptive fashion [1,4,35,44]. For the Stokes eigenvalue problem, Han et al. [21] used mini-element and Sun et al. [41] adopted the Crouzeix-Raviart element and the enriched Crouzeix-Raviart element to discuss the posteriori error estimate based on the multigrid discretization in an adaptive fashion, and its convergence order