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Variable Exponent Herz-Morrey-Hardy Spaces Characterized by Wavelets and Its Application

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Abstract. In this paper, using the atomic decomposition of the Herz-Morrey-Hardy spaces with variable exponent, the wavelet characterization by means of a local version of the discrete tent spaces with variable exponent is established. As an application, the boundedness of the fractional integral operators from variable exponent Herz-Morrey-Hardy spaces into variable exponent Herz-Morrey spaces is obtained.

Key Words: Wavelet, variable exponent, characterization, Herz-Morrey-Hardy space.

AMS Subject Classifications: 42B30, 42B20

1 Introduction

The theory of Hardy spaces constitutes the important part of harmonic analysis. Various characterizations of spaces of functions or distributions, including the atomic and the molecular characterizations, as well as the characterizations by maximal functions, and their applications were studied extensively in harmonic analysis, (see, e.g., [1,3,10–12, 21,25,27]). It is well-known that wavelets with compact support, smoothness or proper decay can be widely applied to characterize function spaces and to construct bases of them (see, e.g., [14,17,22,28,34,35]). In [28], Lu and Yang established some real variable characterizations for Herz-Hardy spaces in terms of the φ-nontangetial maximal function and the φ-radial maximal function. Hernández et al. [13] used compactly supported wavelets in $C^1(\mathbb{R}^n)$, and came up with the φ-transform and wavelet characterizations of Herz-Hardy spaces by means of a local version of the discrete tent spaces at the origin.

On the other hand, Kovčik and J. Rákosník [19] established the Lebesgue and Sobolev spaces with variable exponent as a new method for dealing with nonlinear Dirichlet boundary value problem. Then, differential equation with variable exponent is also intensively developed. Consequently, Diening [7] gave a necessary condition for the continuity of Hardy-Littlewood maximal operator M on Musielak-Orlicz spaces $L^{\varphi}(\mathbb{R}^d)$, and

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showed that the condition is also sufficient in the special case of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$. After that, the theory of function spaces with variable exponent and its applications has made rapidly progress in the past twenty years. In the following, we make a brief view of some important results.

Wang and Liu [33,34] introduced the Herz-Hardy spaces with variable exponent, and generalized some results of [13] and [28]. In fact, the authors obtained the atomic and wavelet characterizations of Herz-Hardy spaces with variable exponent. In 2010, the class of Herz-Morrey spaces with variable exponent was initially defined [16], the authors obtained the boundedness of fractional integrals on Herz-Morrey spaces with variable exponent. Recently, Xu and Yang [35] introduced the Herz-Morrey-Hardy spaces with variable exponent and established the characterization of the spaces in terms of the atoms.

Thus, a natural extension is to consider wavelet characterization of the Herz-Morrey-Hardy spaces with variable exponent. In addition, Kopaliani and Izuki gave the wavelet inequalities of Lebesgue spaces with variable exponent in [14] and [20]. Hence, inspired by the aforementioned references, we focus on the Herz-Morrey-Hardy spaces to establish the wavelet characterization of Herz-Morrey-Hardy spaces with variable exponent. In fact, the decompositions of function spaces make the linear operators acting on spaces very simple (see, e.g., [8, 9, 23, 24, 26, 32, 36]). Since the wavelet characterization of Herz-Morrey-Hardy spaces with variable exponent is obtained, as an application, we proved that the fractional integral operators are bounded from variable exponent Herz-Morrey-Hardy spaces into variable exponent Herz-Morrey spaces.

The organization of this article is as follows. In Section 2, we recall some fundamental facts on the variable exponent functions $p(\cdot)$ and briefly review some standard notations. In Section 3, we introduce a kind of discrete tent spaces with variable exponent and establish characterization of the tent spaces with variable exponent in terms of atoms. In Section 4, we give the wavelet characterization of the Herz-Morrey-Hardy spaces with variable exponent by using the atomic decomposition theory, which is the main result of the present paper. Finally, in Section 5, we obtain the boundedness of the fractional integral operators from variable exponent Herz-Morrey-Hardy spaces into variable exponent Herz-Morrey spaces.

Throughout this paper, we denote the Lebesgue measure, and the characteristic function for a measurable set $E \subset \mathbb{R}^n$ by $|E|$ and χ_E, respectively. C denotes the positive constant which is independent of the main parameters involved, but the value may change from line to line. We also use the notation $a \lesssim b$, if there exists a constant $C > 0$ such that $a \leq Cb$. By $a \approx b$, we mean that $a \lesssim b$ and $b \lesssim a$.

2 Preliminaries

A measurable function $p(x), x \in \mathbb{R}^n$ is said to be a variable exponent if $0 < p(x) < \infty$. Denote by $\mathcal{P}(\mathbb{R}^n)$ to be the set of all variable exponents $p(\cdot)$ such that