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A Spectral Method for a Fokker-Planck Equation in Neuroscience with Applications in Neuron Networks with Learning Rules

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Abstract. In this work, we consider the Fokker-Planck equation of the Nonlinear Noisy Leaky Integrate-and-Fire (NNLIF) model for neuron networks. Due to the firing events of neurons at the microscopic level, this Fokker-Planck equation contains dynamic boundary conditions involving specific internal points. To efficiently solve this problem and explore the properties of the unknown, we construct a flexible numerical scheme for the Fokker-Planck equation in the framework of spectral methods that can accurately handle the dynamic boundary condition. This numerical scheme is stable with suitable choices of test function spaces, and asymptotic preserving, and it is easily extendable to variant models with multiple time scales. We also present extensive numerical examples to verify the scheme properties, including order of convergence and time efficiency, and explore unique properties of the model, including blow-up phenomena for the NNLIF model and learning and discriminative properties for the NNLIF model with learning rules.

AMS subject classifications: 35Q92, 65M70, 92B20

Key words: Integrate-and-Fire model, Fokker-Planck equation, neuron network, spectral methods.

1 Introduction

In recent years, there has been a growing interest in studying large-scale neuron network models, e.g. [3,13,24,27], placing a greater emphasis on the proper mathematical tools for analyzing and simulating the dynamics of such networks. The underlying formulation of

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these models is based on deterministic or stochastic differential equations which describe the activities of neuron ensembles.

In this article, we consider the Nonlinear Noisy Leaky Integrate-and-Fire (NNLIF) model, which was originally proposed in the pioneering works [1,2], and it is one of the fundamental models in computational neuroscience. In the microscopic perspective, this model takes the membrane potential v of neurons as the state variable, which is restricted by a given threshold value V_F [23,25,27,32]. A defining characteristic of this model is the inclusion of firing events, which are described by a reset mechanism: when the membrane potential v reaches the threshold value of V_F, a spike occurs, and the membrane potential is then reset to a lower value V_R. Moreover, the neurons within an ensemble interact with each other only through spikes. In the macroscopic perspective, this model is related to the Fokker-Planck equation [21,22,24], as follows:

$$\begin{cases} \partial_t p+\partial_v(hp)-a\partial_{vv}p=0, \quad v\in(-\infty,V_F]/\{V_R\}, \\ p(v,0)=p^0(v), \quad p(-\infty,t)=p(V_F,t)=0, \\ p(V_R^-,t)=p(V_R^+,t), \quad \partial_v p(V_R^-,t)=\partial_v p(V_R^+,t)+\frac{N(t)}{a}, \end{cases} \tag{1.1}$$

where the probability density function $p(v,t)$ represents the probability of finding a neuron at voltage v and given time t, and $p^0(v)$ is the initial condition. The spiking behavior is described by the mean firing rate $N(t)$, which is implicitly given by

$$N(t)=-a(N(t))\frac{\partial p}{\partial v}(V_F,t). \tag{1.2}$$

The drift coefficient h and diffusion coefficient a are typically expressed as functions of the mean firing rate $N(t)$

$$h(v,N)=-v+bN, \quad a(N)=a_0+a_1N, \tag{1.3}$$

where $-v$ models the leaky mechanism and b represents the connectivity of the network: $b>0$ for excitatory-average networks and $b<0$ for inhibitory-average networks. The connectivity of the network has an essential effect on the properties of (1.1), such as its steady states and blow-up phenomenon. Besides, a stands for the amplitude of the noise, where $a_0>0$ and $a_1\geq 0$. The probability density function $p(v,t)$ should satisfy the condition of conservation of mass

$$\int_{-\infty}^{V_F} p(v,t)dv=\int_{-\infty}^{V_F} p^0(v)dv=1. \tag{1.4}$$

In recent years, there have been significant progress in the numerical and analytical studies of the NNLIF models. [6,13,14] analyze the stability and asymptotic behavior from the point of view of microscopic stochastic differential equation (SDE). From the macroscopic perspective, [3,11,12] establish the existence theory of the Fokker-Planck equation (1.1), and the classical solution exists only when the firing rate N(t) does not diverge. In [3], the authors analyze the model's steady states and blow-up phenomenon. In [16], aiming at