

On the Viability of Solutions to Conformable Stochastic Differential Equations

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Abstract. The viability of the conformable stochastic differential equations is studied. Some necessary and sufficient conditions in terms of the distance function to K are given. In addition, when the boundary of K is sufficiently smooth, our necessary and sufficient conditions can reduce to two relations just on the boundary of K . Lastly, an example is given to illustrate our main results.

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1 Introduction

Fractional derivative is as old as calculus. It is the natural generalization of the ordinary calculus involving derivatives and integrals of noninteger order. For the last few decades, fractional calculus has attracted much attention due to its powerful and widely used tool for better modelling and control of processes in various fields of science, physics, finance, engineering and optimal problem, see [1–3]. Nowadays, there are several definitions of fractional derivatives and integrals such as Riemann-Liouville, Grunwald-Letnikov, Caputo, Weyl [4, 5], Caputo-Fabrizio [6] and Atangana-Baleanu [7]. The most popular definitions are the Riemann-Liouville and Caputo definitions. All definitions of fractional derivatives satisfy the property of linearity. However, almost all fractional derivatives lack the properties of the product rule, quotient rule, chain rule, Rolle's theorem, mean value theorem and composition rule and so on. Due to the special characteristics of the fractional derivative, the compatibility of the stochastic integral and fractional integral encounters many difficulties and limitations.

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To avoid these difficulties, conformable fractional derivative was proposed in Khalil et al. [8]. It has attracted the interest of researchers, as it seems to satisfy all the requirements of the standard derivative. Also, the computing using this new derivative is much easier than using other definitions of fractional derivative. Therefore, there is a large number of works carried out using this new definition and its generalization. The details of the basic theory are reported in [9,10] and the application are reported in [11,12]. The conformable stochastic differential equations were proposed in [13]. It generalized the classical stochastic differential equation and improved the fractional stochastic differential equation. And since the conformable fractional derivative has no special non-local characteristic for the fractional derivative, we can directly express the solution of the equation and calculate the numerical solution, and estimate the error of the asymptotic solution. The Itô formula was established and the existence, uniqueness, continuous dependence and the stability of solutions to the conformable stochastic differential equations were studied in [13,14]. Existence and Stability of Solutions to Neutral Conformable Stochastic Functional Differential Equations were studied in [15].

Given a closed convex set $K \subset \mathbb{R}^n$ and a family X of n -dimensional stochastic processes, one is often interested in the viability of the set K with respect to the family X , that is, for each starting point $x \in K$ the process stays in K . Viability of stochastic systems is an important tool and method to study the comparison theorem and attractor of solutions of stochastic systems. It has important applications in the study of asymptotic stability of stochastic differential equations and synchronous control of systems. The first stochastic viability results can be found in Friedman [16] and Doss [17]. Since then, the viability of the classical stochastic differential equation has been studied extensively. One can refer to the results in [18–32], etc. Up to now, to the best of the author's knowledge, the viability of the conformable stochastic differential equations has not been studied in the literature.

In this paper, we will consider the viability of the following conformable stochastic differential equations

$$\begin{cases} D_\rho^\alpha X(t) = b(X(t), t) + \sigma(X(t), t) \frac{dW(t)}{dt}, & \rho \in (0, 1], t \in [\alpha, \infty), \\ X(\alpha) = X_\alpha, \end{cases} \quad (1.1)$$

where D_ρ^α is conformable derivative, $b: \mathbb{R}^n \times [\alpha, \alpha + h] \rightarrow \mathbb{R}^n$, and $\sigma: \mathbb{R}^n \times [\alpha, \alpha + h] \rightarrow \mathbb{R}^{m \times n}$, $W(t)$ is a standard Wiener process on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, we devote to discussing the necessary and sufficient conditions of the viability of Eq. (1.1), and give some remarks and corollaries. An example is given to illustrate our main results in the final Section.