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AN A PRIORI ERROR ANALYSIS OF A PROBLEM INVOLVING MIXTURES OF CONTINUA WITH GRADIENT ENRICHMENT

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Abstract. In this work, we study a strain gradient problem involving mixtures. The variational formulation is written as a first-order in time coupled system of parabolic variational equations. An existence and uniqueness result is recalled. Then, we introduce a fully discrete approximation by using the finite element method and the implicit Euler scheme. A discrete stability property and a priori error estimates are proved. Finally, some one- and two-dimensional numerical simulations are performed.

Key words. Mixtures, strain gradient, finite elements, discrete energy decay, a priori error estimates, numerical simulations.

1. Introduction

We refer as a mixture of materials to the combination of two or more solids and/or fluids. It is very usual to find mixture of materials in our daily life as far we can use them in the chemical industry or in steel manufacturing (among others). It is common to consider them in the creation of composites which combine several materials with different chemical or physical properties. The main aim is to obtain a new issue which satisfies new specific properties.

To describe these materials it was considered the continuum theory of mixtures. It has become an important field of work for physics, engineers and mathematicians. A mathematical perspective of this theory suggests a relevant family of problems concerning systems of partial differential equations and/or integro-differential equations. It is worth recalling that the current formulation of this theory can be found in the contributions of Bowen and Wise [8], Eringen and Ingram [11, 22], Green and Naghdi [14, 15] and Truesdell and Toupin [29]. Books and classical references on this theory are the works of Atkin and Craine [5, 4], Bedford and Drumheller [6] and Bowen [7]. This theory is totally accepted in the scientific community and it has been extended to consider viscous effects on the different constituents and/or the whole mixture. Some studies concerning these materials can be found in [24, 25, 16, 17, 19, 21, 20], but they are only a few examples in the huge quantity of contributions in this theory.

It is of interest (from a mathematical point of view) to clarify the qualitative and quantitative properties of the solutions to the systems of the differential equations describing mixture of materials. From a qualitative perspective, it is relevant to clarify the existence, uniqueness, continuous dependence and the asymptotic behavior of the solutions. We can cite several contributions [1, 3, 2, 12, 13, 26, 27, 28, 18] in this line.

In this paper, we center our attention in the strain gradient theory of mixtures proposed by Ieşan [18] and we consider several dissipative mechanisms on the conservative structure. It is worth recalling that, from a mathematical perspective, the

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strain gradient theories provide a fourth order spatial derivative in the system of equations and it will be of high interest to clarify the consequences of these components there. We will consider a cylinder of constant cross-section and we will study the behavior of the anti-plane shear deformations. The qualitative study of this problem can be found in [23]. In this new paper, we want to make a numerical contribution to the same problem.

In the next section we describe the mathematical model, we state the basic assumptions to obtain the well-posedness of the problem and we recall an existence and uniqueness result as well as an energy decay property. In Section 3 a fully discrete approximation is introduced by using the finite element method and the implicit Euler scheme. A discrete stability property is proved and a priori error estimates are obtained. Finally, in Section 4 some numerical simulations, involving examples in one and two dimensions, are presented to demonstrate the accuracy of the approximation, the decay of the discrete energy and the behavior of the solution.

2. The basic equations and the variational formulation

In this work, we consider a mixture of two interacting materials. Our domain will consist of one cylinder R of constant cross-section, $R = B \times [0, L]$, where B is a bounded two-dimensional region whose boundary, ∂B, is a curve assumed smooth enough to allow the application of the divergence theorem.

Following [23], we consider the isotropic and homogeneous case for anti-plane shear deformations. It means that we impose the following conditions on the displacements of the two interacting continua $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{w} = (w_1, w_2, w_3)$:

$$u_1 = u_2 = w_1 = w_2 = 0, \quad u_3 = u(x_1, x_2), \quad w_3 = w(x_1, x_2),$$

where u and w are two-dimensional functions which define the displacements of the two constituents in the domain B.

The general problem modeling the evolution of the mixtures with some dissipation mechanisms is written as follows:

$$\left.\begin{aligned} \rho_1 \ddot{u} = &\ \mu_1 \Delta u + \mu \Delta w - \gamma_1 \Delta^2 u - \gamma \Delta^2 w - a(u - w) - \gamma^* \Delta^2 \dot{u} \\ &-a^*(\dot{u} - \dot{w}) + \mu^* \Delta \dot{u}, \\ \rho_2 \ddot{w} = &\ \mu \Delta u + \mu_2 \Delta w - \gamma \Delta^2 u - \gamma_2 \Delta^2 w + a(u - w) + a^*(\dot{u} - \dot{w}), \end{aligned}\right\} \tag{1}$$

where Δ is the two-dimensional Laplacian operator.

In the previous system of equations, we have assumed three possible dissipation mechanisms which correspond to the hyperviscosity (if $\gamma^* \neq 0$ and $a^* = \mu^* = 0$), the weak viscosity (if $a^* \neq 0$ and $\gamma^* = \mu^* = 0$), and the viscosity (if $\mu^* \neq 0$ and $a^* = \gamma^* = 0$). Anyway, we assume that a^*, γ^*, $\mu^* \geq 0$.

As usual in this context, ρ_1 and ρ_2 are assumed to be positive because they represent mass densities. Moreover, in order to guarantee the elastic stability of the materials we will suppose that

$$\mu_1 \mu_2 > \mu^2, \quad \gamma_1 \gamma_2 > \gamma^2, \quad a,\ \mu_1,\ \gamma_1 > 0. \tag{2}$$

We assume also that $\gamma \neq 0$ to ensure the coupling between the materials.

To have a well-posed problem we need to introduce initial and boundary conditions. As initial conditions we consider:

$$\begin{aligned} u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \dot{u}(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in B, \\ w(\boldsymbol{x}, 0) = w_0(\boldsymbol{x}), \quad \dot{w}(\boldsymbol{x}, 0) = e_0(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in B, \end{aligned} \tag{3}$$