INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 21, Number 2, Pages 244–267

doi: 10.4208/ijnam2024-1010

TWO DECOUPLED AND LINEARIZED BLOCK-CENTERED FINITE DIFFERENCE METHODS FOR THE NONLINEAR SYMMETRIC REGULARIZED LONG WAVE EQUATION

JIE XU, SHUSEN XIE, AND HONGFEI FU*

Abstract. In this paper, by introducing a new flux variable, two decoupled and linearized block-centered finite difference methods are developed and analyzed for the nonlinear symmetric regularized long wave equation, where the two-step backward difference formula and Crank-Nicolson temporal discretization combined with linear extrapolation technique are employed. Under a reasonable time stepsize ratio restriction, i.e., $\Delta t = o(h^{1/4})$, second-order convergence for both the primal variable and its flux are rigorously proved on general non-uniform spatial grids. Moreover, based upon the convergence results and inverse estimate, stability of two methods are also demonstrated. Ample numerical experiments are presented to confirm the theoretical analysis.

Key words. Symmetric regularized long wave equation, backward difference formula, Crank-Nicolson, block-centered finite difference method, error estimates.

1. Introduction

Decades ago, numerical simulations of the mathematical models to explain the behavior of nonlinear wave phenomena began to become one of the important scientific research fields. Many nonlinear wave systems are usually used to demonstrate some typical physical problems, such as heat flow phenomena, wave and shallow water wave propagation, optical fiber, hydrodynamics, plasma physics, chemical kinematics, electricity, biology and quantum mechanics [2, 3, 7, 20, 29].

As one of the wave models, the symmetric regularized long wave (SRLW) equation can describe various nonlinear phenomena. The first research result devoted to this model was published by Seyler and Fenstermacher [27] for describing the propagation of ion acoustic waves, shallow water waves, and solitary waves with bidirectional propagation:

$$u_t - u_{xxt} + \rho_x + uu_x = 0, \tag{1}$$

$$\rho_t + u_x = 0, \tag{2}$$

for $(x,t) \in Q = I \times J := (a,b) \times (0,T]$, where ρ and u are dimensionless electron charge density and the fluid velocity, respectively.

In this paper, we are interested to propose two decoupled and linearized finite difference methods on general *non-uniform* spatial grids for (1)–(2) enclosed with the following boundary and initial conditions

$$u_x(a,t) = u_x(b,t) = \rho(a,t) = \rho(b,t) = 0, \quad t \in J, \tag{3}$$

$$u(x,0) = u^o(x), \quad \rho(x,0) = \rho^o(x), \quad x \in \bar{I}, \tag{4}$$

where $u_0(x)$ and $\rho_0(x)$ are two given smooth functions.

Up to now, there are a great amount of work devoting to the traveling wave solution and numerical simulations of the SRLW equation. Existence and uniqueness

Received by the editors on October 11, 2022 and, accepted on January 16, 2024.
2000 *Mathematics Subject Classification.* 65H10, 65M06, 65M12.
*Corresponding author.

of the solution, existence of global attractors, stability and instability of solitary waves and exact traveling wave solution were studied in Refs. [5, 6, 9, 32]. However, the analytical solution of model (1)–(2) is usually not available on bounded domain. Therefore, efficient numerical methods and numerical analysis are necessary, of which the finite difference method is viewed as one of the most significant numerical methods. In Ref. [31], Wang, Zhang and Chen proposed three nonlinear and linear three-level second-order difference schemes on *uniform* spatial grids, in which two are coupled and one is decoupled. The convergence estimates of the approximate solutions to u and ρ were proved to be $\mathcal{O}(\Delta t^2 + h^2)$ in discrete L^∞ and L^2 norm, respectively. Li [19] studied a conservative weighted compact difference method on *uniform* grids as well, and proved that the convergence order is $\mathcal{O}(\Delta t^2 + h^4)$ in discrete L^∞ norm for u and L^2 norm for ρ, respectively. Recently, He et al. [13] constructed a fourth-order accurate compact difference scheme for the SRLW equation, and they also analyzed the convergence and stability of the scheme, but only for u in discrete L^∞ norm with a *uniform* spatial grids. Besides, In Ref. [14], He, Wang and Dai developed two dissipative difference schemes for the generalized SRLW equations, where one is a two-level nonlinear coupled scheme and the other is a three-level linear decoupled scheme. Convergence order $\mathcal{O}(\Delta t^2 + h^4)$ on *uniform* spatial grids and stability in discrete L^∞ norm for u and L^2 norm for ρ were proved by the discrete energy method. Dirichlet boundary conditions are involved in all papers mentioned above. There are also some other numerical methods for the SRLW model, see finite difference methods [4, 12, 16, 21, 24, 36], spectral and pseudo spectral methods [10, 18, 28, 37, 38], and finite element methods [11, 22, 23, 34]. However, as far as we know, there are still no papers concerning finite difference methods on general *non-uniform* meshes.

In real simulations of the nonlinear SRLW equation, the flux of the primal variable usually represents the velocity variation, and sometimes it is of great importance to calculate the flux in high-order accuracy, which is also the motivation of our concern on space discritization. It is well known that block-centered finite difference (BCFD) method can simultaneously approximate the primal variable and its flux to a same order of accuracy on non-uniform grids without any accuracy lost, compared to the standard finite difference method. It can be thought of as the lowest-order Raviart-Thomas mixed element method [26] by employing a proper numerical quadrature formula. Thus, the method is widely studied in the literature. For example, Weiser and Wheeler [30] studied the BCFD method for elliptic problems with Neumann boundary conditions in one and two-dimensional cases. They demonstrated that with sufficiently smooth data, the discrete L^2-norm errors for both the approximate solution and its first derivatives are in second-order for all non-uniform grids. In Ref. [1], Arbogast, Wheeler and Yotov presented the mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences. Besides, in Refs. [17, 25, 35], some BCFD methods were developed to solve flow models such as the multiscale flows model, Darcy-Forchheimer model, and semiconductor device model. Basically, second-order spatial convergence were observed therein. In summary, the BCFD method could keep second-order spatial accuracy both for the original unknown, called pressure in porous media flow, and its derivatives, called velocity in porous media flow, on general non-uniform spatial grids. Thus, it is widely used even for problems with boundary layers and large gradient deformations.

As far as we know, there seems to be no published work on BCFD method for the nonlinear SRLW equation with Neumann boundary conditions. Our main goal