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Two-Grid Methods for Maxwell's Equations in a Cole-Cole Dispersive Medium

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Abstract. A two-grid method (TGM) for the time-dependent Maxwell's equations in Cole-Cole dispersive media with a fractional time derivative term is proposed. We employ the lowest Raviart-Thomas-Nédélec mixed finite elements to discrete the space. It is known that for these type of Nédélec edge finite elements, the standard TGM cannot be applied directly. Therefore, we modified the traditional TGM, and the discrete process can be divided into two steps. Firstly, we get the rough discrete solutions on the coarse mesh. At the same time, superconvergence results can be obtained by using a post-processing technique. Secondly, the superconvergent solutions on the coarse grid are added on the fine mesh as a correction, and the optimal error estimates could be obtained accordingly. Finally, the numerical experiments can verify that the theoretical results are correct and reasonable.

AMS subject classifications: 35R11, 65N30, 78M10

Key words: Maxwell's equations, two-grid method, Raviart-Thomas-Nédélec mixed finite elements, postprocessing technique.

1. Introduction

There are a variety examples of dispersion media, including plasma, water, biological tissue and radar absorbing material. Apart from typical models such as Lorentz and Debye models, K.S. Cole and R.H. Cole [8] demonstrated experimental results indicating that a so-called Cole-Cole dispersion relation can accurately characterize the broadband dielectric response for dispersive materials.

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In this paper, we discuss the following Maxwell's equations in Cole-Cole dispersive media:

$$\epsilon_0\epsilon_\infty\frac{\partial \boldsymbol{E}}{\partial t} = \nabla\times\boldsymbol{H} - \frac{\partial \boldsymbol{P}}{\partial t}, \qquad (\boldsymbol{x},t)\in\Omega\times(0,T], \tag{1.1}$$

$$\mu_0\frac{\partial \boldsymbol{H}}{\partial t} = -\nabla\times\boldsymbol{E}, \qquad (\boldsymbol{x},t)\in\Omega\times(0,T], \tag{1.2}$$

$$\tau_0^\alpha\mathscr{D}_t^\alpha\boldsymbol{P} + \boldsymbol{P} = \epsilon_0(\epsilon_s-\epsilon_\infty)\boldsymbol{E}, \quad (\boldsymbol{x},t)\in\Omega\times(0,T], \tag{1.3}$$

$$\boldsymbol{n}\times\boldsymbol{E} = 0, \qquad (\boldsymbol{x},t)\in\partial\Omega\times(0,T], \tag{1.4}$$

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}), \quad \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(\boldsymbol{x}), \quad \boldsymbol{P}(\boldsymbol{x},0) = \boldsymbol{P}_0(\boldsymbol{x}) = 0, \quad \boldsymbol{x}\in\Omega, \tag{1.5}$$

where Ω is a bounded and convex Lipschitz polyhedral domain, $\boldsymbol{E}, \boldsymbol{P}$, and $\boldsymbol{H}$ respectively denote the electric field, the induced polarization field, and magnetic field — cf. [19]. Besides, μ_0, ϵ_∞, τ_0, ϵ_s, ϵ_0 are given physical constants, which are the permeability of free space, the infinite-frequency permittivity, the relaxation time, the static relative permittivity, the permittivity of free space, respectively. Here $\mathscr{D}_t^\alpha\boldsymbol{P}(t)$ is defined by the Caputo fractional derivative [19], viz.

$$\mathscr{D}_t^\alpha\boldsymbol{P}(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\boldsymbol{P}_s(s)ds, \quad \alpha\in(0,1).$$

The problem (1.1)-(1.3) is equipped with the boundary condition (1.4) and initial conditions (1.5), where $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are given functions.

In effect, these types of Cole-Cole relation models have applications in many scientific fields — e.g. polymers [12,39], biological media [17], and the frequency electrical response of porous rocks [33]. Compared with the classical frequency dependent dispersive media models — e.g. plasma, Debye, Lorentz, and Drude models, the superiority of Cole-Cole models is that these time-domain polarization relations involve fractional derivative terms which can better simulate the interactions between microwave pulses and biological tissues [10]. Hence the study of numerical solutions for Cole-Cole models has attracted much attention.

The finite-difference time-domain (FDTD) method has been extensively used for describing wave propagation in a Cole-Cole model [32, 34, 38]. In general, two main strategies for such problems are classified according to FDTD schemes. In the first approach, the Cole-Cole dispersion relationship was transformed into a convolution integral which represents the relations among the electric field with polarization current, and a sum of decaying exponentials has been adopted to approximate a convolution integral [38]. Following this method, Schuster *et al*. [34] obtained a faster decayed time series and used it for approximating the convolution integral. In the second approach, a new system is formed by adding an auxiliary differential equation (ADE) to the original Maxwell's curl equations, in which the ADE is obtained by transforming the polarization equation from the frequency to the time domain through inverse Fourier transform. Compared to the former approach with complex convolution counterpart, the application of ADE based approach is quite simple, direct and effective. By using Padé approximation, Rekanos *et al*. [32] transformed