INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, Series B
Volume 4, Number 3, Pages 215–223

ON THE CACCIOPPOLI INEQUALITY OF THE UNSTEADY STOKES SYSTEM

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Abstract. In this paper we study on a Caccioppoli inequality of the unsteady Stokes system.

Key words. Cacciappoli inequality, nonsteady Stokes system.

1. Introduction

Let Ω be an n-dimensional domain, $n = 3$. For $x_0 \in \mathbb{R}^n$, we set $B_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$ and $S_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| = r\}$. For the Laplace equation

$$-\Delta u = 0 \text{ in } \Omega,$$

or for the steady Stokes system

$$-\Delta u + \nabla p = 0, \text{ div } u = 0 \text{ in } \Omega,$$

the following inequality

$$\|\nabla u\|^2_{L^2(B_r(x_0))} \leq \frac{C}{(\rho - r)^2} \|u\|^2_{L^2(B_\rho(x_0))} \tag{1.1}$$

holds for any $0 < r < \rho$ with $C_1 r \leq \rho < \text{dist}(x_0, \partial\Omega)$ for some $C_1 > 1$, which is called the Caccioppoli inequality. Caccioppoli inequality is very important tool for the regularity estimate of elliptic partial differential equations (see [4, 3] and references therein.)

Let ϕ be a cut-off function with $\phi = 1$ in $B_r(x_0)$ and $\phi = 0$ exterior to $B_\rho(x_0)$. Caccioppoli inequality for the Laplace equation is easily obtained by testing $u\phi^2$ to the both side of Laplace equation. On the other hand, testing $u\phi^2$ to the steady Stokes system we have

$$\|\nabla u\|^2_{L^2(B_r(x_0))} \leq \frac{C}{(\rho - r)^2} \|u\|^2_{L^2(B_\rho(x_0))} + \frac{C}{\rho - r} \|p - \bar{p}_r\|_{L^2(B_\rho(x_0))} \|u\|_{L^2(B_\rho(x_0))},$$

where $\bar{p}_\rho = \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} p(y)dy$. There is ψ satisfying

$$\text{div}\psi = p - \bar{p}_\rho \text{ in } B_\rho(x_0), \psi = 0 \text{ on } S_\rho(x_0),$$

$$\|\nabla \psi\|_{L^2(B_\rho(x_0))} \leq C \|p - \bar{p}_\rho\|_{L^2(B_\rho(x_0))}$$

(see [1, 2] and references therein). Testing ψ to the steady Stokes system we have

$$\|p - \bar{p}_\rho\|_{L^2(B_\rho(x_0))} \leq C \|\nabla u\|_{L^2(B_\rho(x_0))}.$$

Now, by the standard argument such as in Giaquinta[4] we have the Cacciopili inequality for the steady Stokes system.

Received by the editors April 10, 2013 and, in revised form, June 6, 2013.

2000 *Mathematics Subject Classification.* 35K51,76D07.

The author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology NRF 2011-0007701.

Caccioppoli inequality for a parabolic partial differential equations should be as follows:

$$\sup_{t_0-r^2<t<t_0} \|u(t)\|_{L^2(B_r(x_0))} + \|\nabla u\|_{L^2(Q_r(x_0))} \leq \frac{C}{\rho - r}\|u\|_{L^2(Q_\rho(x_0))}$$

for any $0 < r < \rho$ with $C_1 r \leq \rho < \min\{\text{dist}(x_0, \partial\Omega), \sqrt{t_0}\}$ for some $C_1 > 1$, where $Q_r(x_0) = B_r(x_0) \times (t_0 - r^2, t_0)$ for $x_0 \in \Omega$ and $t_0 > 0$. For the heat equation the Caccioppoli inequality can be obtained by testing $u\phi^2$ as before.

In this paper we consider (u, p) satisfying the unsteady Stokes system

$$\partial_t u - \Delta u + \nabla p = 0, \ \text{div} u = 0 \text{ in } \Omega \times (0, \infty). \tag{1.2}$$

Here $\partial_t = \frac{\partial}{\partial t}$.

Unlike to the steady Stokes system or the heat equation Caccioppoli inequality for the unsteady Stokes system has not been known well. The main difficulty lies on the fact that for the unsteady Stokes system p cannot be treated separately with $\partial_t u$. Once Cacciappoli inequality would be given, regularity estimate of Navier-Stokes system could be proceeded. In [5], the following Cacciappoli type inequality

$$\|\nabla^2 u\|_{L^2(Q_r(x_0))} \leq \frac{C}{(\rho - r)^2}\|u\|_{L^2(Q_\rho(x_0))}$$

has been shown for the unsteady Stokes system, by decomposing pressure p, and it has been used for the regularity estimate of a weak solution of the Navier-Stokes system.

Our goal is to derive a Caccioppoli inequality for the unsteady Stokes system. For the purpose of it, we introduce a vector field v so that $\nabla \times (\phi v)$ is comparable to $\phi^2 u$. Testing $\nabla \times (\phi v)$ to the unsteady Stokes system, the term p can be disregarded.

The following is our main result.

Theorem 1.1. *Let $x_0 \in \Omega$ and $t_0 > 0$, where $\Omega \subset \mathbb{R}^n$, $n \geq 3$. Take $0 < r < \rho$ with*

$$C_1 r \leq \rho < \min\{\sqrt{t_0}, \textit{dist}(x_0, \partial\Omega)\} \textit{ for some } C_1 > 0.$$

Let (u, p) be a weak solution of the unsteady Stokes system (1.2). *Then*

$$\|\nabla u\|_{L^2(Q_r(x_0))} \leq \frac{C}{\rho - r}\|u\|_{L^2(Q_\rho(x_0))}. \tag{1.3}$$

We also derive the Caccioppoli inequality for the higher derivatives.

Theorem 1.2. *Under the same assumptions as in Theorem 1.1 for r, ρ, x_0, t_0, and (u, p), we also have*

$$\sup_{t_0-r^2<t<t_0} \|\nabla \times u(t)\|_{L^2(B_r(x_0))} + \|\nabla^2 u\|_{L^2(Q_r(x_0))} \leq \frac{C}{(\rho - r)^2}\|u\|_{L^2(Q_\rho(x_0))}. \tag{1.4}$$

In section 4 we also derive Caccioppoli type inequality for further terms(see Theorem 4.1). For the proof of Theorem 1.1, Theorem 1.2 and Theorem 4.1, we remark that the interior regularity of the weak solution of the Stokes system is well known

Throughout this paper, all the constant C depend only on n(and independent of r.) We denote by $\|\cdot\|_{L^q}$ the norm $\|\cdot\|_{L^q(\mathbb{R}^n)}$. We denote by B_r and Q_r the sets $B_r = B_r(x_0)$ and $Q_r = B_r \times (-r^2 + t_0, t_0]$.