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AN EXPERIMENTAL STUDY OF SEVERAL MULTIDIMENSIONAL, LOCALLY CONSERVATIVE, EULERIAN-LAGRANGIAN FINITE ELEMENT METHODS FOR A SEMILINEAR PARABOLIC EQUATION

SON-YOUNG YI AND JIM DOUGLAS, JR.

Abstract. This paper is an experimental continuation of [7], where we presented one realization of a locally-conservative Eulerian-Lagrangian finite element method (LCELM) for a semilinear parabolic equation and proved an optimal convergence rate. In this paper, we present two higher-order extensions of the method of [7], along with one lower-order procedure. We show some numerical results to illustrate the accuracy and efficiency of the LCELM procedures. Optimal convergence rates for each method will be presented.

Key words. locally conservative, Eulerian-Lagrangian, semilinear parabolic equation, finite element method

1. Introduction.

A large number of physical problems of significant interest are described by convection-dominated diffusive systems. Such problems occur in multiphase and multicomponent flows in porous media coming from petroleum and environmental engineering, aerodynamics, and the modeling of semiconductor devices. It has been recognized for several decades that standard finite difference and finite element methods often lead to inaccurate approximations of the underlying phenomena. In response to this, a variety of numerical techniques have been introduced, with many classified as Eulerian-Lagrangian methods, in which an Eulerian finite difference or finite element treatment of diffusion is combined with a Lagrangian treatment of convection. Some methods fail to conserve the material or mass in the system, and others may preserve mass globally (in space) at all time levels but not locally in space. There are, however, a number of very important physical problems for which local conservation is essential. In order to achieve this local conservation of mass, a method called the Locally Conservative Eulerian-Lagrangian Method (LCELM) was introduced for the problem of two-phase, immiscible, incompressible flow in porous media [5]. A reasonably extensive set of computational experiments were presented to validate the new method and to show that it produces a more detailed picture of the local behavior in waterflooding a fractally heterogeneous medium. A convergence analysis has been given for the LCELM as it relates to a single space variable semilinear parabolic problem [4]. More recently, in [7], we presented one realization of the LCELM as applied to a scalar semilinear parabolic problem in two space variables and proved it to be convergent at an optimal rate as the spatial and temporal discretizations are refined. The main object here is to consider several higher-order extensions of the method of [7], along with one lower-order procedure; No formal convergence proofs have been obtained for these methods, but we shall present computations to indicate that these procedures do lead to optimal order convergence.

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The rest of the paper is organized as follows: in Section 2, we describe a semilinear parabolic equation that is of interest in this paper and operator splitting procedures that the LCELM is based on. Section 3 introduces the lowest-order version of the LCELM and its post-processed version that employ the lowest-order Raviart-Thomas space. In Section 4, we describe two higher-order extensions of the post-post processed version of the method. One employs the Brezzi-Douglas-Fortin-Marini space of index 2 to achieve second-order accuracy in space. Another is a Crank-Nicolson version of the method. Finally, in Section 5 we conduct numerical experiments to observe the convergence behavior and mass conservation property of each version of LCELM and compare the LCELM with the original modified method of characteristics (MMOC) [6].

2. A Locally Conservative Eulerian-Lagrangian Method

We shall treat the same differential problem (1) as in [7]. Let Ω be a bounded domain in R^2 and consider the following initial-boundary value problem for a semilinear parabolic equation:

$$\nabla_{t,\mathbf{x}} \cdot \begin{pmatrix} s(\mathbf{x},t) \\ f(s)\mathbf{u}(\mathbf{x},t) \end{pmatrix} - \nabla_{\mathbf{x}} \cdot (d(\mathbf{x})\nabla_{\mathbf{x}} s) = g(\mathbf{x},t), \qquad \mathbf{x} \in \Omega,\ t \in (0,T], \tag{1a}$$

$$(f(s)\mathbf{u}(\mathbf{x},t) - d(\mathbf{x})\nabla_{\mathbf{x}} s) \cdot \mathbf{n} = 0, \qquad \mathbf{x} \in \partial\Omega, \tag{1b}$$

$$s(\mathbf{x},0) = s_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{1c}$$

where $\mathbf{u}$ is known and satisfies the equation

$$\nabla_{\mathbf{x}} \cdot \mathbf{u}(\mathbf{x},t) = b(\mathbf{x},t), \quad t > 0. \tag{1d}$$

Let N_x, N_y, and N_t be positive integers. We define a uniform space-time partition on $\Omega \times [0,T]$ by

$$\begin{aligned} &x_{i_1} = i_1\Delta x, && i_1 = 0,1,\cdots,N_x, && \Delta x = 1/N_x, \\ &y_{i_2} = i_2\Delta y, && i_2 = 0,1,\cdots,N_y, && \Delta y = 1/N_y, \\ &t^n = n\Delta t, && n = 0,1,\cdots,N_t, && \Delta t = T/N_t, \end{aligned}$$

and let $h = \max\{\Delta x, \Delta y\}$.

The locally conservative, Eulerian-Lagrangian methods considered herein are based on the following operator-splitting procedure to separate transport from diffusion:

1° **Initialize:**

$$S^0(\mathbf{x}) = s_0(\mathbf{x}),\ \mathbf{x} \in \Omega. \tag{2a}$$

2° **Transport:** for $1 \le n \le N_t$ and $t^{n-1} < t \le t^n$,

$$\nabla_{t,\mathbf{x}} \cdot \begin{pmatrix} \overline{S} \\ f(\overline{S})\mathbf{u} \end{pmatrix} = g, \tag{2b}$$

$$\overline{S}(\mathbf{x},t^{n-1}) = S^{n-1}(\mathbf{x}). \tag{2c}$$

3° **Diffusion:** for $1 \le n \le N_t$ and $t^{n-1} < t \le t^n$,

$$\frac{\partial \widehat{S}}{\partial t} - \nabla_{\mathbf{x}} \cdot (d\nabla_{\mathbf{x}}\widehat{S}) = 0, \tag{2d}$$

$$\widehat{S}(\mathbf{x},t^{n-1}) = \overline{S}(\mathbf{x},t^n). \tag{2e}$$