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COMPUTATION OF STATIONARY PULSE SOLUTIONS OF THE CUBIC-QUINTIC COMPLEX GINZBURG-LANDAU EQUATION BY A PERTURBATION-INCREMENTAL METHOD

Y.Y. CAO AND K.W. CHUNG

Abstract. Stationary pulse solutions of the cubic-quintic complex Ginzburg-Landau equation are related to heteroclinic orbits in a three-dimensional dynamical systems and they are usually obtained using numerical simulation. The harmonic balance method has severe limitation in computing homoclinic/heteroclinic orbits since the period of such orbits is infinite. In this paper, we present a perturbation-incremental method to find such stationary pulse solutions. With the introduction of a nonlinear transformation, perturbed analytical pulse solutions are obtained in terms of trigonometric functions. Such formulation makes it possible to apply the harmonic balance method to find accurate approximate solutions of the corresponding heteroclinic orbits with arbitrary parametric values. Zero-order analytical solutions from the perturbation step and approximate solutions from the incremental step are compared with that from the bifurcation package AUTO, and they are in good agreement.

Key words. Cubic-quintic complex Ginzburg-Landau equation, homoclinic/heteroclinic orbit, perturbation-incremental method, pulse.

1. Introduction

The complex Ginzburg–Landau equation (CGLE) has a long history as a generic amplitude equation derived asymptotically near the onset of instabilities in fluid dynamical systems. It models the formation of patterns in nonlinear dissipative media, with important applications in physics and chemistry [26, 1] such as superconductivity, superfluidity, Bose-Einstein condensation to liquid crystals and strings in field theory [14, 7, 19, 3, 8, 23]. It comes from the nonlinear Schrödinger equation with the inclusion of the growth and damping terms. A rich variety of behavior such as self-replicating [21, 11] and elastic behavior upon collision [13, 27] are observed from its coherent structures such as pulses and holes. For the cubic CGLE, exact stationary Pereira-Stenflo soliton was obtained in [22] for arbitrary growth and damping strength. It has also a one-parameter family of exact traveling Nozaki-Bekki holes [20, 24] which are structurally unstable. The cubic-quintic CGLE is known to admit stable coherent structures as a consequence of the coexistence between a stable limit cycle and a stable fixed point and its nonvariational nature. Exact solutions are available only when the system parameters of the cubic-quintic CGLE satisfy certain conditions. To investigate the behavior and bifurcation of coherent structure for arbitrary parametric values, numerical simulation or continuation starting from an exact solution is usually employed [10, 28, 17]. However, continuation based on the harmonic balance (HB) method using Fourier series has not been employed to study coherent structures of the CGLE although the HB method is an efficient technique in computing limit cycles of dynamical systems [15, 16, 18, 12]. It is due to the fact that exact solutions of CGLE are usually expressed in terms of Jacobi elliptic functions or hyperbolic functions.

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Correspondence author: K. W. Chung (makchung@cityu.edu.hk).

On the other hand, coherent structures of the CGLE are related to homoclinic/heteroclinic orbits of a three-dimensional system of ordinary differential equations. For small perturbation, Belhaq et al [2] applied the elliptic Lindstedt-Poincaré method to determine an approximation of the limit cycles near homoclinicity. They imposed a criterion based on a collision between limit cycles with large period and the saddle point. Chen & Chen [5] presented a hyperbolic perturbation method using similar collision criterion. For large parametric values, the HB method has severe limitation in computing homoclinic/heteroclinic orbits since the period of such orbits is infinite. In numerical computation, not only it requires a large number of harmonic terms but also the error is rather significant [29].

In [9], Descalzi proposed an analytical method which consisted of calculating localized solutions inside and outside the core and then to match the approximate solutions at the border of the regions, imposing there continuity of the amplitude, the phase, and the derivative of the amplitude. The method is able to predict the range of existence of local structures. However, the bifurcation curve obtained from the above method has obvious deviation from that using the bifurcation package AUTO.

The present study is motivated from the above situation. Recently, we proposed a novel construction of homoclinic/heteroclinic orbits for planar nonlinear oscillators using a nonlinear time transformation [4]. Even for large parametric values, only a few harmonic terms can achieve high accuracy. In [6], the novel construction is applied to find exact front, pulse and hole solutions for a two-dimensional cubic CGLE in terms of Fourier series. An advantage of this formulation is that perturbed analytical solution can be obtained near an exact solution. Continuation based on the HB method becomes possible in finding and analyzing coherent structures of the cubic-quintic CGLE for arbitrary parametric values.

The paper is organized as follows. In Section 2, exact stationary pulse and hole solutions of the cubic CGLE are obtained in terms of trigonometric functions by using a nonlinear time transformation. Such transformation is applied in Section 3 to find perturbed analytical pulse solutions of the cubic-quintic CGLE from which the range of existence of pulses can be predicted. In Section 4, a continuation procedure based on the HB method is proposed to compute pulse solutions for arbitrary parametric values, which are compared with that from AUTO. A conclusion is given in Section 5.

2. Stationary coherent structure of the cubic CGLE

In this section, we first consider the cubic CGLE in the following form

$$\partial_t A = \mu A + \beta |A|^2 A + D\partial_{xx} A. \tag{1}$$

We are going to find exact stationary pulse and hole solutions of (1) using the nonlinear transform φ introduced in [4], and compare with those from current literature. Suppose the ansatz

$$A = \mathrm{e}^{\mathrm{i}(\theta(\xi)+\omega t)} u(\xi), \tag{2}$$

where $\theta(\xi)$ and $u(\xi)$ are real functions with $\xi = x - vt$. Substituting (2) into (1) and after simplification, we obtain

$$u_{\xi\xi} - u\Psi^2 + vD_1 u_\xi + vD_2 u\Psi + \mu_1 u + \beta_1 u^3 = 0, \tag{3a}$$

$$2u_\xi \Psi + u\Psi_\xi + vD_1 u\Psi - vD_2 u_\xi - \mu_2 u + \beta_2 u^3 = 0, \tag{3b}$$