INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 2, Number 1, Pages 27–41

THE FINEST LEVEL ACCELERATION OF MULTILEVEL AGGREGATION FOR MARKOV CHAINS

CHUN WEN, TING-ZHU HUANG, DE-AN WU, AND LIANG LI

Abstract. In this paper, we consider a class of new accelerated multilevel aggregation methods using two polynomial-type vector extrapolation methods, namely the reduced rank extrapolation (RRE) and the generalization of quadratic extrapolation (GQE) methods. We show how to combine the multilevel aggregation methods with the RRE and GQE algorithms on the finest level in order to speed up the numerical computation of the stationary probability vector for an irreducible Markov chain. Numerical experiments on typical Markov chain problems are reported to illustrate the efficiency of the accelerated multilevel aggregation methods.

Key words. Markov chains, multilevel aggregation, acceleration, vector extrapolation methods.

1. Introduction

The use of Markov chains is of interest in a wide range of applications, including information retrieval and web ranking [12, 28, 29, 32], queueing systems [16] and stochastic automata networks [14], as well as performance modeling of computer and communication systems, dependability and security analysis, and analysis of biological systems [40].

In this paper, we study a class of new accelerated multilevel aggregation methods which are efficient for computation of the stationary probability vector of an irreducible Markov chain. Mathematically, the problem to be solved is given by

$$Bx = x, \quad x_i \geq 0 \ \forall i, \quad ||x||_1 = 1, \tag{1}$$

where $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ is a column stochastic matrix, i.e., $0 \leq b_{ij} \leq 1 \ \forall i, j$ and $e^{\mathrm{T}} B = e^{\mathrm{T}}$, with e the column vector with all elements as one, and $x \in \mathbb{R}^n$ is the stationary probability vector of Markov chains. In fact, if B is irreducible [5, 40], which is equivalent to the existence of a directed path from any vertex i to any other vertex j in the directed graph of B, then x is the unique solution to the linear system (1). Moreover, the stationary probability vector x of the Markov chains satisfies the inequality $x_i > 0 \ \forall i$.

For convenience, we rewrite the equation (1) as

$$Ax = 0, \tag{2}$$

with

$$A = I - B, \tag{3}$$

where I is an identity matrix, and A is a singular M-matrix with diagonal elements being the negative column sums of its off-diagonal elements. Hence, our interest is

Received by the editors June 21,2010 and, in revised form, November 30 ,2010 .
2000 *Mathematics Subject Classification.* 65C40, 60J22.
The authors would like to express their great thankfulness to the anonymous referees and Professor Yau Shu Wong for their constructive and helpful comments leading to the improvement of the paper. This research was supported by 973 Program (2007CB311002), NSFC (60973015), Sichuan Province Sci. & Tech. Research Project (2009SPT-1, 2009GZ0004).
This research was supported by NSFC (10926190, 60973015) and 973 Program (2007CB311002).

to solve the $n \times n$ homogeneous linear system (2) corresponding to an irreducible Markov chain.

Iterative procedures are commonly used numerical methods to compute the stationary probability vector for an irreducible Markov chain. Examples of iterative techniques include the power methods for calculating the dominant eigenvector of the matrix B [26–28, 33, 38], the Gauss-Seidel, SOR and SSOR iteration methods based on splitting of the matrix A [33, 34], the iterative aggregation/disaggregation algorithms for Markov chains [30, 31, 36, 41], the hybrid algorithm for queueing systems [48], and the well-known Krylov subspace methods such as the Arnoldi's algorithms [23, 45], BiCGSTAB and GMRES methods [33–35].

However, the iteration methods are likely to suffer from a slow convergence for some linear systems, for example, computing the principal eigenvector for Google matrix. Thus it is necessary to employ the idea of preconditioning. Philippe, Saad and Stewart considered three different incomplete factorizations: ILU0, ILUTH and ILUK as preconditioners for numerical solutions of Markov chain modelling in [33]. Virnik presented an algebraic multigrid preconditioner for M-matrices in [44]. Benzi and Uçar developed the block triangular and product preconditioners based on the alternating iteration [1] for M-matrices and the Markov chain problems in [2] and [3], respectively. In addition, the applications of circulant preconditioners for Markov chains had been report in [14, 15].

Recently, multilevel methods based on aggregation of the Markov states have been studied in the literature [17–20, 24, 25]. Isensee and Horton considered multi-level methods for the steady state solution of a continuous-time (CTMC) and discrete-time (DTMC) Markov chains in [25] and [24], respectively. De Sterck, Manteuffel, Mccormick, Nguyen and Ruge proposed a multilevel adaptive aggregation (MAA) method to calculate the stationary probability vector of Markov matrices in [18]. As already showed in [18], the multilevel method is a special case of the adaptive smoothed aggregation [8] and adaptive algebraic multigrid methods [7] for sparse linear systems.

Thereafter, De Sterck et al. proposed several strategies to accelerate the convergence of the multilevel aggregation methods. Such strategies include the application of a smoothing technique to the interpolation and restriction operators [17], and analyzing a recursive accelerated multilevel aggregation method by computing quadratic programming problems with inequality constraints[20]. In particular, a top-level acceleration of adaptive algebraic multilevel method was considered by finding a linear combination of previous fine-level iterates, so that it minimizes a functional over a subset of the probability vectors in [19]. The active set method from matlab's quadprog function [22] was used in their implementations.

Here, we consider a class of new accelerated multilevel aggregation methods by the use of two polynomial-type vector extrapolation methods: the reduced rank extrapolation (RRE) and the generalization of quadratic extrapolation (GQE) methods proposed by Sidi[38]. In fact, the idea to improve iteration methods by combining with vector extrapolation methods is not new; see [10, 11, 28, 39, 45]. This paper shows how to combine the multilevel aggregation methods with RRE and GQE algorithms on the finest level in order to speed up the numerical calculation of the stationary probability vector for an irreducible Markov chain. In numerical experiments, the accelerated multilevel aggregation methods are tested using three representative Markov chain problems. The problems include the nearly completely decomposable (NCD) Markov chains, which are difficult to solve since they consist