INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 2, Number 4, Pages 338–354

LOCALIZED POINTWISE ERROR ESTIMATES AND GLOBAL L^p ERROR ESTIMATES FOR NITSCHE'S METHOD

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Abstract. We derive localized pointwise error estimates for Nitsche's method applied to an elliptic second order problem in $\mathbf{R}^n$ $(n = 2, 3)$. Using these results, we also prove quasi-optimal global L^p error estimates as well. Numerical experiments are provided which back up the theoretical findings.

Key words. Nitsche's method, pointwise error estimates, L^p error estimates

1.1. Introduction.

We consider the following second order elliptic problem:

$$\mathcal{L}u := -\nabla \cdot (\mathbf{A}\nabla u) + \mathbf{b} \cdot \nabla u + cu = f \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1.1b}$$

Here, $\Omega \subset \mathbf{R}^n$ $(n = 2, 3)$ is an open bounded domain with smooth boundary, $\mathbf{b} \in \mathbf{R}^n$, $c \in \mathbf{R}$, and $\mathbf{A} \in \mathbf{R}^{n\times n}$ is a symmetric positive definite matrix. More assumptions about the data and domain are given in the following subsection.

Recall that Nitsche's method [17] for (1.1) is defined as seeking a function $u_h \in V_h$ such that

$$\begin{aligned} A(u_h, v) := &\int_\Omega \Big(\big(\mathbf{A}\nabla u_h\big) \cdot \nabla v + \mathbf{b} \cdot \nabla u_h v + cuv \Big)\, dx + \eta \sum_{e\in\mathcal{E}_h^b} \frac{1}{h_e} \int_e u_h v \, ds \\ &- \sum_{e\in\mathcal{E}_h^b} \int_e \big(\mathbf{A}\nabla u_h\big) \cdot n_e v\, ds - \sum_{e\in\mathcal{E}_h^b} \int_e \big(\mathbf{A}\nabla v\big) \cdot n_e u_h \, ds = \int_\Omega fv\, dx \quad \forall v \in V_h, \end{aligned} \tag{1.2}$$

where V_h is the finite element space, and η denotes the penalty parameter which imposes the boundary conditions (1.1b) weakly into the variational formulation (a detailed description of the notation used above is presented in the following subsections). It is well-known that if the penalty parameter is taken sufficiently large then the method (1.2) is well-posed (cf. Lemma 2.1 below). Moreover, due to Lemma 2.2 and Céa's Lemma [3, 6], we have

$$\|u - u_h\|_{W_h^{1,2}(\Omega)} \le C \inf_{v\in V_h} \|u - v\|_{W_h^{1,2}(\Omega)}, \tag{1.3}$$

where $\|\cdot\|_{W_h^{1,2}(\Omega)}$ denotes a special energy norm defined below, and C denotes a generic positive constant. The goal of this paper is to derive localized pointwise and global L^p $(2 \le p \le \infty)$ error estimates for Nitsche's method (1.2). One motivation to derive such estimates is its use in the convergence analysis of a fully nonlinear problem [16].

Many contributions have been made to establish pointwise and L^p estimates for classical finite element methods for problems such as (1.1), and we mention the most significant results. Broadly speaking, the analysis of pointwise estimates

Received by the editors November 8, 2010 and, in revised form, November 23, 2011.
2000 *Mathematics Subject Classification.* 65M60, 65N30.
This work was supported by the National Science Foundation under grants DMS-0902683 and DMS-1115421.

can be divided into two groups. The first group, which started with the work of Natterer, Nitsche, Scott, Frehse, and Rannacher [15, 9, 23, 21, 22, 3] obtained L^p bounds with the use of certain type of *weighted L^1 estimates* for discrete Green functions. Except for the recent result found in [3, Corollary 8.2.8], all of these estimates are global, that is, the error at a certain point depends equally on the smoothness of u on all of the domain [12]. In particular, all of these estimates have the form

$$\|u - u_h\|_{L^p(\Omega)} \le Ch|\ln h|^{\bar{k}} \inf_{v \in V_h} \|u - v\|_{W^{1,p}(\Omega)}, \tag{1.4a}$$

$$\|u - u_h\|_{W^{1,p}(\Omega)} \le C \inf_{v \in V_h} \|u - v\|_{W^{1,p}(\Omega)}, \tag{1.4b}$$

where $p \in [2, \infty]$, k denotes the polynomial degree of the finite element space, and $\bar{k} = 1$ if $k = 1$ and $p = \infty$, and $\bar{k} = 0$ otherwise.

In contrast, the second group, which started with the work of Schatz [19] and later extended by various authors [11, 8, 14], used *local L^1 error estimates* of an auxiliary discrete Green function to derive weighted pointwise estimates. Using such techniques, they were able to derive estimates at an arbitrary point which depends strongly on u only near the point, namely,

$$|(u - u_h)(z)| \le Ch|\ln h|^{\bar{s}} \inf_{v \in V_h} \|u - v\|_{W^{1,\infty}(\Omega),z,s} \qquad 0 \le s \le k-1, \tag{1.5a}$$

$$|\nabla(u - u_h)(z)| \le C|\ln h|^{\bar{\bar{s}}} \inf_{v \in V_h} \|u - v\|_{W^{1,\infty}(\Omega),z,s} \qquad 0 \le s \le k. \tag{1.5b}$$

Here, $\bar{s} = 1$ if $s = k-1$, $\bar{s} = 0$ if $s < k-1$, $\bar{\bar{s}} = 1$ if $s = k$, $\bar{\bar{s}} = 0$ if $s < k$, and $\|\cdot\|_{W^{1,\infty}(\Omega),z,s}$ is a weighted norm concentrated at the point z with strength s. In addition to being sharper than (1.4) in the case $p = \infty$, estimates such as (1.5) and the techniques to derive them spawned new applications such as asymptotic error expansion inequalities and new *a posteriori* residual type estimators [19, 8, 7].

In the context of discontinuous Galerkin (DG) methods, there have also been many contributions to develop L^∞ error estimates by different authors [13, 4, 5, 11]. The first by Kanschat and Rannacher [13], which generalizes earlier work of Rannacher, Frehse and Scott, used a duality argument and weighted L^1 estimates of a discrete Green function. As a result, they obtained estimates of the form (1.4) in the case $p = \infty$ and piecewise linear polynomials are used. This work was later extended and improved by Chen and Chen [4] who derived localized pointwise error estimates similar to (1.5) using techniques developed by Schatz.

Since Nitsche's method can be considered a discontinuous Galerkin method restricted to the continuous Lagrange finite element space, it seems plausible that the analysis for pointwise estimates for DG methods can be used and adapted for Nitsche's method. This is the approach we take. Since the sharpest results of pointwise estimates for DG methods were achieved by Chen and Chen, we follow their approach and derive pointwise error estimates that are similar to (1.5). As expected, most of the analysis found in [4] carries over to the case in hand. However, an added feature in our analysis is that we also derive global L^p error estimates using the analysis and techniques to derive the pointwise estimates.

1.2. Organization of Paper. In the following subsection, we set the notation that will be used throughout the paper and then state our main results, Theorems 1.1–1.3. The rest of the paper is devoted to proving these Theorems. First, in Section 2 we state some preliminary estimates that were shown in [19, 4, 17, 2, 6] which are used frequently in the main proofs. With this completed, in Section 3 we show that proving the pointwise estimates stated in Theorem 1.1 reduces to