INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 2, Number 4, Pages 379–401

REMARK ON STABILITY OF TRAVELING WAVES FOR NONLOCAL FISHER-KPP EQUATIONS

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Abstract. This paper is concerned with a class of nonlocal Fisher-KPP type reaction-diffusion equations in n-dimensional space with time-delay. It is proved that, all noncritical planar wavefronts are exponentially stable in the form of $t^{-\frac{n}{2}}e^{-\nu_\tau t}$ for some constant $\nu_\tau = \nu(\tau) > 0$, where $\tau \geq 0$ is the time-delay, while the critical planar wavefronts are algebraically stable in the form of $t^{-\frac{n}{2}}$. These convergent rates are optimal in the sense with L^1-initial perturbation. The adopted approach is the weighted energy method combining Fourier transform. It is also realized that, the effect of time-delay essentially causes the decay rate of the solution slowly down. These results significantly generalize and develop the existing study [37] for 1-D time-delayed Fisher-KPP type reaction-diffusion equations. When the time-delay τ vanishes, we automatically obtain the exponential stability for the noncritical planar traveling waves and the algebraic stability for the critical planar traveling waves to the regular Fisher-KPP equations.

Key words. Nonlocal reaction-diffusion equations, time delays, traveling waves, global stability, the Fisher-KPP equation, L^1-weighted energy, Green functions.

1. Introduction and Main Results

Following the recent study [37] on the stability of traveling waves to 1-D nonlocal time-delayed reaction-diffusion equations, in this paper, we study a class of n-D nonlocal Fisher-KPP reaction-diffusion equations ([4, 11, 25, 37])

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u + d(u) = \displaystyle\int_{\mathbf{R}^n} f_\alpha(y) b(u(t-\tau, x-y))dy, \\ u|_{t=s} = u_0(s,x), \ x \in \mathbf{R}^n, s \in [-\tau, 0] \end{cases} \tag{1}$$

for $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$ and $t \geq 0$. Here, $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, $D > 0$ is the diffusion coefficient, $\tau \geq 0$ is the time-delay, $f_\alpha(y)$, with $\alpha > 0$, is the heat kernel in the form of

$$f_\alpha(y) = \frac{1}{(4\pi\alpha)^{\frac{n}{2}}} e^{\frac{-|y|^2}{4\alpha}} \quad \text{with} \quad \int_{\mathbf{R}^n} f_\alpha(y)dy = 1, \tag{2}$$

$d(u)$ and $b(u)$ both are nonlinear functions satisfying

- (H_1) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = 0$, $d(u_+) = b(u_+)$, and $d(u), b(u) \in C^2[0, u_+]$;
- (H_2) $b'(0) > d'(0) \geq 0$ and $0 \leq b'(u_+) < d'(u_+)$;
- (H_3) For $0 \leq u \leq u_+$, $d'(u) \geq 0$, $b'(u) \geq 0$, $d''(u) \geq 0$, $b''(u) \leq 0$.

The model of (1) describes the wave propagations in fluid dynamics, and in physical, chemical and biological dynamics, initially given by R.A. Fisher [10], and A. Kolmogoroff, I. Petrovsky and N. Piscounoff [22]. The study on such a wave propagation phenomenon can be also found in [1, 31] for the fluid dynamical experiments on Taylor-Couette flow, in [7] for Rayleigh-Benard flow, in [44, 52] for the

Received by the editors June 11, 2011.
2000 *Mathematics Subject Classification.* 35K57, 34K20, 92D25.
This research was supported by the NSERC of Canada.

chemical wave experiments, and in [3] for population dynamics, combustion, and biological invasions.

In the equation (1), if we take $\tau = 0$ and $\alpha \to 0^+$, and use the property of heat kernel $f_\alpha(y)$:

$$b(u(t,x)) = \lim_{\alpha\to 0^+} \int_{\mathbf{R}^n} f_\alpha(y) b(u(t,x-y))dy, \tag{3}$$

we derive the following regular Fisher-KPP reaction-diffusion equation [3, 10, 9, 15, 53, 55]

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u = h(u), \\ u|_{t=0} = u_0(x), \ x \in \mathbf{R}^n, \end{cases} \tag{4}$$

with $h(u) = b(u) - d(u)$. Particularly, taking $d(u) = u^2$ and $b(u) = u$, then we reduce (4) to the following classical Fisher-KPP equation [3, 8, 10, 12, 21, 22, 41, 43]

$$\frac{\partial u}{\partial t} - D\Delta u = u(1-u), \quad t > 0, \, x \in \mathbf{R}^n. \tag{5}$$

Clearly, from (H_1), both $u_- = 0$ and $u_+ > 0$ are constant equilibria of the equation (1); and from (H_2), $u_- = 0$ is unstable and u_+ is stable for the spatially homogeneous equation associated with (1); and from (H_3), both $b(u)$ and $d(u)$ are increasing, and $b(u)$ is concave downward and $d(u)$ is concave upward. These characters let the equations (1) and (4) capture the most basic features of the classical Fisher-KPP equation (5), so we call the equations (1) and (4) as the nonlocal/local Fisher-KPP type reaction-diffusion equations. Except the standard example with $b(u) = u$ and $d(u) = u^2$ for the classical Fisher-KPP equation (5), equation (1) includes the other two important examples. One is the Nicholson's blowflies equation [27, 28, 30, 35, 36, 37, 38, 39, 47, 48]

$$\frac{\partial u}{\partial t} - D\Delta u + \delta u(t,x) = \varepsilon p \int_{\mathbf{R}^n} f_\alpha(y) u(t-\tau, x-y) e^{au(t-\tau,x-y)} dy,$$

with

$$b(u) = \varepsilon p u e^{-au} \ \text{ and } \ d(u) = \delta u, \quad \varepsilon > 0, \ p > 0, \ a > 0, \ \delta > 0.$$

Obviously, these specified functions $b(u)$ and $d(u)$ satisfy (H_1)-(H_3) with $u_- = 0$ and $u_+ = \frac{1}{a} \ln \frac{\varepsilon p}{\delta}$ for $1 < \frac{\varepsilon p}{\delta} \le e$. The other typical example is the age-structured population model [2, 13, 14, 26, 37, 40]

$$\frac{\partial u}{\partial t} - D\Delta u + \delta u^2(t,x) = p e^{-\gamma\tau} \int_{\mathbf{R}^n} f_\alpha(y) u(t-\tau, x-y) dy,$$

with

$$d(u) = \delta u^2 \ \text{ and } \ b(u) = p e^{-\gamma\tau} u, \quad \delta > 0, \ p > 0, \ \gamma > 0,$$

which also satisfy (H_1)-(H_3) automatically with $u_- = 0$ and $u_+ = \frac{p}{\delta} e^{-\gamma\tau}$.

A *planar traveling wavefront* to the equation (1) is a special solution in the form of $u(t,x) = \phi(x \cdot \mathbf{e} + ct)$ with $\phi(\pm\infty) = u_\pm$, where c is the wave speed, $\mathbf{e}$ is a unit vector of the basis of $\mathbf{R}^n$. Without loss of generality, we can always assume $\mathbf{e} = \mathbf{e}_1 = (1, 0, \cdots, 0)$ by rotating the coordinates. Thus, we have the planar traveling wavefront in the form $\phi(x \cdot \mathbf{e}_1 + ct) = \phi(x_1 + ct)$, which satisfies, for $\tau \ge 0$,

$$\begin{cases} c\phi' - D\phi'' + d(\phi) = \displaystyle\int_{\mathbf{R}^n} f_\alpha(y) b(\phi(\xi_1 - y_1 - c\tau)) dy, \\ \phi(\pm\infty) = u_\pm, \end{cases} \tag{6}$$