INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 1, Pages 48–62

ANALYSIS OF A MIXED-SHEAR-PROJECTED QUADRILATERAL ELEMENT METHOD FOR REISSNER-MINDLIN PLATES

GUOZHU YU, XIAOPING XIE*, AND YUANHUI GUO

Abstract. This paper analyzes an existing 4-node hybrid mixed-shear-projected quadrilateral element MiSP4, presented by Ayad, Dhatt and Batoz (Int. J. Numer. Meth. Engng 1998, 42: 1149-1179) for Reissner-Mindlin plates, which behaves robustly in numerical benchmark tests. This method is based on Hellinger-Reissner variational principle, where continuous piecewise isoparametric bilinear interpolations, as well as the mixed shear interpolation/projection technique of MITC family, are used for the approximations of displacements, and piecewise-independent equilibrium modes are used for the approximations of bending moments/shear stresses. Due to local elimination of the parameters of moments/stresses, the computational cost of MiSP4 element is almost the same as that of the conforming bilinear quadrilateral displacement element. We show that the element is free from shear locking in the sense that the error bound in the derived a priori estimate is independent of the plate thickness.

Key words. Reissner-Mindlin plate, mixed-shear-projected quadrilateral element, shear-locking free.

1. Introduction

Due to avoidance of C^1-continuity difficulty, the Reissner-Mindlin (R-M) plate model is today the dominating two-dimensional model used to calculate the bending of a thick/thin three-dimensional plate of thickness t. It's well-known that for values of t close to zero, the standard low-order finite element discretization of this model suffers from shear locking ([1, 17]).

To overcome the shear locking difficulty and derive 'locking-free' or robust plate bending elements that are valid for the analysis of thick and thin plates, significant efforts are devoted to the development of simple and efficient triangular and quadrilateral finite elements in the past few decades. The most common approach is to modify the variational formulation with some reduction operator so as to weaken the Kirchhoff constraint (see, e.g. [2]-[8], [10], [12], [14]-[16], [18]-[20] and the references therein).

Among the existing elements, the family of finite elements named mixed interpolated tensorial components (MITC) by Bathe et. al [4, 5] is one of the most attractive representative. By virtue of an independent shear approximation and a discrete Mindlin technique along edges, MITC elements define the shear strains in terms of the edge tangential strains that are projected on the element degrees of freedom. As the lowest order quadrilateral MITC element, the 4-node plate element MITC4 is very likely the most used in practice.

Using the same technique of shear interpolation as in the element MITC family, Ayad, Dhatt and Batoz [3] presented an improved formulation for obtaining locking-free quadrilateral element, which is called MiSP4 element. It is based on Hellinger-Reissner variational principle, including variables of displacements, shear

Received by the editors on October 25, 2015.
2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.
*Corresponding author.

stresses and bending moments. For the approximations of displacements, MiSP4 element uses continuous piecewise isoparametric bilinear interpolations. For the approximations of bending moments/shear stresses, it uses piecewise-independent equilibrium modes. The numerical experiments are presented to show that the MiSP4 element can avoid locking phenomenon, and it also passes the patch test for a general quadrilateral. However, so far there is no uniform error analysis with respect to plate thickness. It should be pointed out that in a very recent paper [8], the shear interpolation treatment was replaced by enhancing a shear-stress-enhanced condition, and the resultant 4-node hybrid finite element scheme was shown to be locking-free.

The main goal of this work is to establish uniform convergence for quadrilateral MiSP4 element. The main tools of our analysis are the self-equilibrium relation, i.e. (21), which contributes to the uniform coercivity of the corresponding bilinear forms, and the properties of shear interpolation proved in [11] for MITC4 element (see Lemma 4.11).

We arrange the rest of this paper as follows. In Section 2 we give weak formulation of the model. Section 3 introduces the finite element spaces for MiSP4 element. We derive in Section 4 uniform error estimates for MiSP4 element. Finally in Section 5 we provide some numerical results to verify the theoretical results.

For convenience, throughout the paper we use the notation $a \lesssim b$ to represent that there exists a generic positive constant C, independent of the mesh parameter h and the plate thickness t, such that $a \leq Cb$. We also abbreviate $a \lesssim b \lesssim a$ as $a \approx b$.

2. Weak problem

The Reissner-Mindlin model for the bending of a clamped isotropic elastic plate in equilibrium reads as: Find $(w, \boldsymbol{\beta}) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$ such that

$$-\mathbf{div}\mathcal{D}\boldsymbol{\epsilon}(\boldsymbol{\beta}) - \lambda t^{-2}(\text{grad } w - \boldsymbol{\beta}) = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$-\lambda t^{-2}\text{div}(\text{grad } w - \boldsymbol{\beta}) = g \quad \text{in} \quad \Omega. \tag{2}$$

Here $\Omega \subset \mathbb{R}^2$, assumed to be a convex polygon for simplicity, is the region occupied by the midsection of the plate with plate thickness t, w and $\boldsymbol{\beta}$ denote respectively the transverse displacement of the midplane and the rotation of the fibers normal to it, $\boldsymbol{\epsilon}(\boldsymbol{\beta})$ is the symmetric part of the gradient of $\boldsymbol{\beta}$, g is the transverse loading, $\mathcal{D}$ is the elastic module tensor defined by

$$\mathcal{D}\mathbf{Q} = \frac{E}{12(1-\nu^2)}[(1-\nu)\mathbf{Q} + \nu\text{tr}(\mathbf{Q})\mathbf{I}]$$

with $\mathbf{Q}$ a 2×2 symmetric matrix, $\lambda = \frac{\kappa E}{2(1+\nu)}$ with E the Young's modulus, ν the Poisson's ratio, and $\kappa = \frac{5}{6}$ the shear correction factor.

Set

$$\mathbb{M} := L^2(\Omega)^{2\times 2}_{sym}, \quad \Gamma := L^2(\Omega)^2, \quad W := H_0^1(\Omega), \quad \Theta := H_0^1(\Omega)^2.$$

When introducing the shear stress vector $\boldsymbol{\gamma} = \lambda t^{-2}(\text{grad } w - \boldsymbol{\beta})$ and the bending moment tensor $\mathbf{M} = -\mathcal{D}\boldsymbol{\epsilon}(\boldsymbol{\beta})$, the model problem (1)-(2) changes into the following system: Find $(\mathbf{M}, \boldsymbol{\gamma}, w, \boldsymbol{\beta}) \in \mathbb{M} \times \Gamma \times W \times \Theta$ such that

$$\mathbf{divM} - \boldsymbol{\gamma} = 0 \quad \text{in} \quad \Omega, \tag{3}$$

$$\text{div}\boldsymbol{\gamma} + g = 0 \quad \text{in} \quad \Omega, \tag{4}$$

$$\mathbf{M} + \mathcal{D}\boldsymbol{\epsilon}(\boldsymbol{\beta}) = 0 \quad \text{in} \quad \Omega, \tag{5}$$

$$\boldsymbol{\gamma} - \lambda t^{-2}(\text{grad } w - \boldsymbol{\beta}) = 0 \quad \text{in} \quad \Omega. \tag{6}$$