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FLUX RECOVERY AND SUPERCONVERGENCE OF QUADRATIC IMMERSED INTERFACE FINITE ELEMENTS

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Abstract. We introduce a flux recovery scheme for the computed solution of a quadratic immersed finite element method introduced by Lin *et al.* in [13]. The recovery is done at nodes and interface point first and by interpolation at the remaining points. In the case of piecewise constant diffusion coefficient, we show that the end nodes are superconvergence points for both the primary variable p and its flux u. Furthermore, in the case of piecewise constant diffusion coefficient without the absorption term the errors at end nodes and interface point in the approximation of u and p are zero. In the general case, flux error at end nodes and interface point is third order. Numerical results are provided to confirm the theory.

Key words. Recovery technique, quadratic immersed interface method, Superconvergence, conservative method, Green's function.

1. Introduction

We consider the interface two-point boundary value problem

$$\begin{cases} -(\beta(x)p'(x))' + q(x)p(x) = f(x), & x \in (a,b), \\ p(a) = p(b) = 0, \end{cases} \tag{1}$$

where $q(x) \geq 0$ and $0 < \beta \in C(a,\alpha) \cup C(\alpha, b))$ is piecewise constant with a finite jump across the interface point α so that the solution p satisfies

$$[p]_\alpha = 0, \tag{2}$$

$$[\beta p']_\alpha = 0, \tag{3}$$

where $[s]_\alpha = s^+ - s^-$ denotes the jump of the quantity s across α.

Physically the variable p may stand for the pressure or temperature in a material with certain physical properties and the derived quantity $u := -\beta p'$ is the corresponding flux, which may be of equal interest. The piecewise constant β reflects a nonuniform material and the function $q(x)$ reflects a property of the material or its surroundings. In this paper we will refer to p as pressure. Problem (1)-(3) can also be viewed as the steady neutron diffusion problem [14]. Due to its simple structure, a lot of its mathematical and numerical properties of related numerical methods can be explicitly worked out. Therefore, it is very instructive to study this problem before moving to its higher dimensional and/or nonsteady state versions. It is in this spirit that we shall study the immersed finite elements for this problem.

Efficient numerical methods for (1)-(3) may use meshes that are either fitted or unfitted with the interface. A method allowing unfitted meshes would be very efficient when one has to follow a moving interface in a temporal problem. For an in-depth exposition of the numerics and applications of interface problems, we refer the readers to [9] and the references therein. In an immersed finite element (IFE) method, the mesh is made up of interface elements where the interface intersects elements (thus immersed) and noninterface elements where the interface is absent.

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On a noninterface element one uses standard local shape functions, whereas on an interface element one uses piecewise standard local shape functions subject to continuity and jump conditions. Representative works on IFE methods can be found in [1, 7, 8, 9, 10, 11, 12, 16], among others. In particular, reference [1] gives a unified discussion on 1-D IFE spaces, including the quadratic space. A more recent contribution highly related to the present paper is [2] (pointed out by a referee), which studies superconvergence using generalized Legendre polynomials.

In this paper, we are interested in studying IFE methods that can produce accurate approximate flux u_h of p once an approximate p_h has been obtained, particularly those that can recover flux without having to solve a system of equations. Chou and Tang [6] initiated such methods when the mesh is fitted. Later it was generalized to the immersed interface mesh case using linear immersed finite elements (IFE) of Lin *et al.* [13] and their variants for one dimensional elliptic and parabolic problems [4, 5]. In this paper we concentrate on quadratic elements. We aim at a method that will extend good features such as existence of superconvergence points and the discrete conservation law that we have either proved or observed in the linear immerse finite element case.

To begin with, let's first give the central idea [6] behind our flux recovery scheme on a mesh $\{t_i\}$. Suppose we want to evaluate $u(t_i)$ at some mesh point t_i using some weighted integral of p. We can proceed as follows. Let ϕ be a function with compact support K such that $I_i = [t_{i-1}, t_i] \subset K$, the interface point $\alpha \notin K$, $\phi(t_{i-1}) = 0$, $\phi(t_i) = 1$. An example of such a function is the standard finite element hat function. Multiplying (1) by ϕ and integrating by parts, we see that the flux u satisfies

$$u(t_i) = -\int_{I_i} \beta p'\phi' dx - \int_{I_i} qp\phi dx + \int_{I_i} f\phi dx.$$

It is then natural to define an approximate flux u_h at t_i as

$$u_h(t_i) = -\int_{I_i} \beta p_h'\phi' dx - \int_{I_i} qp_h\phi dx + \int_{I_i} f\phi dx.$$

The error $E_i := u(t_i) - u_h(t_i)$ then satisfies

$$E_i = -\int_{I_i} \beta(p' - p_h')\phi' dx - \int_{I_i} q(p - p_h)\phi dx.$$

In the case that ϕ is linear on I_i, $q = 0$, $p = p_h$ at t_{i-1}, t_i, we immediately see that the error in flux is also zero at t_i. With a little calculation using the jump conditions (2)-(3), the same line of thought works when $\alpha \in I_i$. In this paper the ϕ's will be from the immersed quadratic shape functions and we show in Thm 3.2 that in the case of $q = 0$, the quadratic IFE solution $p_h = p$ at all end nodes and as a consequence $u = u_h$ at those points as well. When $q \neq 0$, the exactness cannot be attained due to the nature of the Green's function involved (see the proof of Thm 3.3), but those points are still superconvergence points of the pressure and flux . Another feature of our scheme is that when $q = 0$ the following conservation law or discrete first fundamental theorem of calculus holds:

$$u_h(t_i) - u_h(t_{i-1}) = \int_{I_i} f(x)dx,$$

whose continuous version can be obtained for the exact flux from integrating (1). The above two features in higher dimensional IFE methods are under investigation [3]. Finally, since the IFE reduces to the standard finite elements in absence of the interface, the superconvergence results in this paper also apply to the standard