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A Remark on the Courant-Friedrichs-Lewy Condition in Finite Difference Approach to PDE's

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Abstract. The Courant-Friedrichs-Lewy condition (The CFL condition) is appeared in the analysis of the finite difference method applied to linear hyperbolic partial differential equations. We give a remark on the CFL condition from a view point of stability, and we give some numerical experiments which show instability of numerical solutions even under the CFL condition. We give a mathematical model for rounding errors in order to explain the instability.

AMS subject classifications: 65M06, 65M12, 65G50

Key words: Numerical analysis, finite difference scheme, stability, CFL condition.

1 Introduction

The finite difference method (FDM) has been discussed as one of the mathematical tools to deal with partial differential equations before the era of digital computers till now, and R. Courant, K. O. Friedrichs and H. Lewy gave precise discussion about asymptotic behaviour of FDM solutions in [1]; we can see "we will find that for the case of the initial value problem for hyperbolic equations, convergence is obtained only if the ratio of the mesh widths in different directions satisfies certain inequalities which in turn depend on the position of the characteristics relative to the mesh" in [1] (this sentence is quoted from its English translation [2]). As is pointed out in [6], we note as follows; it is a necessary condition of convergence for FDM solution that the region of dependence of the finite

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difference schema contains that of the corresponding hyperbolic differential equation. We call this condition the Courant-Friedrichs-Lewy condition (CFL condition).

Let us introduce the CFL condition for the case

$$\frac{\partial}{\partial t}u(t,x)=c\frac{\partial}{\partial x}u(t,x), \qquad x\in\mathbb{R}, \quad t>0, \tag{1.1a}$$

$$u(0,x)=u_0(x), \qquad x\in\mathbb{R}, \tag{1.1b}$$

where c is a positive constant. Let Δt and Δx be increments along the t-direction and x-direction respectively, and we give a discretization by FDM

$$\frac{u_j^{k+1}-u_j^k}{\Delta t}=c\frac{u_{j+1}^k-u_j^k}{\Delta x}, \tag{1.2}$$

where u_j^k denotes the value corresponding to $u(k\Delta t,j\Delta x)$. Introduction of $\lambda:=\Delta t/\Delta x$ leads us to

$$u_j^{k+1}=c\lambda u_{j+1}^k+(1-c\lambda)u_j^k, \tag{1.3}$$

and the inequality $1-c\lambda\geq 0$ is the CFL condition for (1.2) and (1.3).

We sometimes encounter misunderstanding that the CFL condition is considered as a stability condition of the schema (1.2) or (1.3). The authors are afraid that it is caused by misunderstanding the proper meaning of the Lax equivalence theorem [5]. The analysis of stability of numerical solutions is one of the most important ones in the theory of FDM, and we may find its origin in the historic paper [7] of G. G. O'Brien et al. They gave, following the fundamental study of von Neumann, definition of stability so as to estimate numerical errors mainly coming from the rounding errors. Lax et al. [5] gave definition of stability as a discrete analogue to well-posedness of differential equations, and they declared that numerical errors were not taken into account in [5]. Lax et al., on the other hand, referred O'Brien et al. [7] as analysis of influence of numerical errors, and the authors are afraid that many researchers may have misunderstood the differences between their analyses.

Proper understanding of the analysis by O'Brien et al. [7] and by Lax et al. [5] is very much significant in order to understand real effects of rounding errors in computation by FDM. The authors consider it to be equivalent to give an answer to the question whether stability conditions guarantee stable computation less influenced by the rounding errors. For clear discussion, we need a mathematical model of propagation of the rounding errors in computation, but we should remark that modeling is dependent upon a system of floating point numbers on computers. When we restrict ourselves on computation of an evolutional finite difference schema, we notice that O'Brien et al. [7] adopted a naive model in their analysis of propagation of the rounding errors, and the authors should remark that it has close relation with stability analysis by Lax et al. [5]. Unfortunately it is inadequate in explanation of severe effects of the rounding errors occurred on *current* digital computers.