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Approximation of the Cubic Functional Equations in Lipschitz Spaces

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Abstract. Let $\mathcal{G}$ be an Abelian group and let $\rho:\mathcal{G}\times\mathcal{G}\to[0,\infty)$ be a metric on $\mathcal{G}$. Let $\mathcal{E}$ be a normed space. We prove that under some conditions if $f:\mathcal{G}\to\mathcal{E}$ is an odd function and $C_x:\mathcal{G}\to\mathcal{E}$ defined by $C_x(y):=2f(x+y)+2f(x-y)+12f(x)-f(2x+y)-f(2x-y)$ is a cubic function for all $x\in\mathcal{G}$, then there exists a cubic function $C:\mathcal{G}\to\mathcal{E}$ such that $f-C$ is Lipschitz. Moreover, we investigate the stability of cubic functional equation $2f(x+y)+2f(x-y)+12f(x)-f(2x+y)-f(2x-y)=0$ on Lipschitz spaces.

Key Words: Cubic functional equation, Lipschitz space, stability.

AMS Subject Classifications: 39B82, 39B52

1 Introduction

Let $\mathcal{G}$ be an abelian group and $\mathcal{E}$ a normed space. Let $S(\mathcal{E})$ be a family of subsets of $\mathcal{E}$. We say that $S(\mathcal{E})$ is linearly invariant if it is closed under the addition and scalar multiplication defined as usual sense and translation invariant, i.e., $x+A\in S(\mathcal{E})$, for every $A\in S(\mathcal{E})$ and every $x\in\mathcal{E}$ (see [2]). It is easy to verify that $S(\mathcal{E})$ contains all singleton subsets of $\mathcal{E}$. In particular, $CB(\mathcal{E})$ the family of all closed balls with center at zero is a linearly invariant family in a normed vector space $\mathcal{E}$. By $F(\mathcal{G},S(\mathcal{E}))$ we denote the family of all functions $f:\mathcal{G}\to\mathcal{E}$ such that $\mathrm{Im}f\subset A$ for some $A\in S(\mathcal{E})$. We say that $F(\mathcal{G},S(\mathcal{E}))$ admits a left invariant mean (briefly LIM), if the family $S(\mathcal{E})$ is linearly invariant and there exists a linear operator $\Gamma:F(\mathcal{G},S(\mathcal{E}))\to\mathcal{E}$ such that

(i) if $\mathrm{Im}f\subset A$ for some $A\in S(\mathcal{E})$, then $\Gamma[f(\cdot)]\in A$,

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(ii) if $f\in F(\mathcal{G},S(\mathcal{E}))$ and $a\in\mathcal{G}$, then $\Gamma[f^a(\cdot)]=\Gamma[f(\cdot)]$,

where $f^a(\cdot)=f(\cdot+a)$ (see [14,20]). Following [3] and [20], let $S(\mathcal{E})$ be a linearly invariant family and $\mathbf{d}:\mathcal{G}\times\mathcal{G}\to S(\mathcal{E})$ be a set-valued function such that

$$\mathbf{d}(x+a,y+a)=\mathbf{d}(a+x,a+y)=\mathbf{d}(x,y),$$

for all $a,x,y\in\mathcal{G}$. A function $f:\mathcal{G}\to\mathcal{E}$ is said to be $\mathbf{d}$-Lipschitz if $f(x)-f(y)\in\mathbf{d}(x,y)$ for all $x,y\in\mathcal{G}$.

Let $(\mathcal{G},\rho)$ be a metric group and $\mathcal{E}$ a normed space. A function $\alpha_f:\mathbb{R}^+\to\mathbb{R}^+$ is a module of continuity of $f:\mathcal{G}\to\mathcal{E}$ if $\rho(x,y)\le r$ implies $\|f(x)-f(y)\|\le\alpha_f(r)$ for every $r>0$ and every $x,y\in\mathcal{G}$. A function $f:\mathcal{G}\to\mathcal{E}$ is called Lipschitz function if it satisfies the condition

$$\|f(x)-f(y)\|\le L\rho(x,y) \tag{1.1}$$

for every $x,y\in\mathcal{G}$. We denote by $\mathsf{lip}(f)$ the smallest constant $L\in\mathbb{R}^+$ satisfying the condition (1.1). By $Lip^0(\mathcal{G},\mathcal{E})$ we denote the normed space of all Lipschitz functions $f:\mathcal{G}\to\mathcal{E}$ with the norm defined by the formula

$$\|f\|_{Lip^0}:=\|f(0)\|+\mathsf{lip}(f).$$

In particular, $Lip(\mathcal{G},\mathcal{E})$ the space of all bounded Lipschitz functions is a subspace of $Lip^0(\mathcal{G},\mathcal{E})$ and with the norm defined by

$$\|f\|_{Lip}:=\|f\|_{\sup}+\mathsf{lip}(f)$$

is another normed space (see [20]).

Ulam [21] in 1940 stated the following problem, called now as the problem of stability of functional equations. Let X be an abelian group and Y an abelian group with a metric d. Let $\epsilon>0$ be given. Does there exist $\lambda>0$ such that if $f:X\to Y$ satisfies

$$d[f(x+y),f(x)+f(y)]<\epsilon$$

for all $x,y\in X$, then there exists an additive function $A:X\to Y$ with

$$d[f(x),A(x)]<\lambda$$

for all $x\in X$?

In the next year D. H. Hyers [16] proved the problem for the Cauchy functional equation. The result of Hyers was generated by Th. M. Rassias [18]. For more details about the results concerning such problems the reader is referred to [1,5,19].

Recently, S. Czerwik and K. Dlutek [4], established the stability of quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

in Lipschitz spaces. In this paper, we investigate the same results for cubic functional equation

$$2f(x+y)+2f(x-y)+12f(x)-f(2x+y)-f(2x-y)=0.$$