INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 12, Number 4, Pages 731–749

VARIATIONAL FORMULATION FOR MAXWELL'S EQUATIONS WITH LORENZ GAUGE: EXISTENCE AND UNIQUENESS OF SOLUTION

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Abstract. The existence and uniqueness of a vector scalar potential representation with the Lorenz gauge (Schelkunoff potential) is proven for any vector field from H(curl). This representation holds for electric and magnetic fields in the case of a piecewise smooth conductivity, permittivity and permeability, for any frequency. A regularized formulation for the magnetic field is obtained for the case when the magnetic permeability μ is constant and thus the magnetic field is divergence free. In the case of a non divergence free electric field, an equation involving scalar and vector potentials is proposed. The solution to both electric and magnetic formulations may be approximated by the nodal shape functions in the finite element method with system matrices that remain well-conditioned for low frequencies. A numerical study of a forward problem of a computation of electromagnetic fields in the diffusive electromagnetic regime shows the efficiency of the proposed method.

Key words. Lorenz gauge, Schelkunoff potential, Maxwell's equations, Finite Element Method, Nodal shape functions, Regularization

1. Introduction

Fast and stable methods are needed for calculating electromagnetic (EM) fields in and over the Earth. Such a simulation has applications in imaging of subsurface electrical conductivity structures related to exploration for geothermal, mining, and hydrocarbon resources. Over commonly used frequencies, EM propagation in the Earth is diffusive since the conduction dominates over the dielectric displacement. The finite element method (FEM) is attractive for this simulation in comparison with other techniques in that it may be easily adapted to complex boundaries between regions of constant EM properties, including the topography or the bathymetry. The 3D interpretation of geophysical data is numerically expensive, as the forward problem needs to be computed many times [26, 3, 14].

For large scale simulation problems, iterative methods have been the ones of choice to solve linear systems resulting from FEM formulations [7, 16, 11, 34, 29]. The speed of iterative methods is strongly related to the properties of the variational problem used. Difficulties arise when the computational domain includes a high contrast, both the non-conducting air and a conducting medium in the Earth's subsurface, especially for low frequencies. Furthermore, the Earth's subsurface in general is characterized by the spatially changing conductivity, dielectric permittivity and magnetic permeability. This can slow or prevent iteration convergence [23, 31].

There have been multiple approaches to addressing the difficulties encountered with high physical property contrasts and potentially discontinuous EM field variables. One is to apply special finite elements, so-called edge elements, that have a discontinuous normal component of the vector field across elements, while keeping the tangential field component continuous [24, 18, 4]. The edge elements are also

Received by the editors June 21, 2013 and, in revised form, January 25, 2015.
2000 *Mathematics Subject Classification.* 65N30,86A04,35Q61.

compatible with the curl operator and are a part of the de Rham diagram [6]. However, if the curl-curl equation for the electric field E is used, and if the conductivity is very small in a part of the domain (e.g., in the air) or if the frequency is very low, the problem becomes ill-posed and the system matrix has a very large near null space. This requires use of sophisticated preconditioners that handle the null space of the curl properly in order to use iterative solvers. Such preconditioners have been developed (see [38, 17, 19, 21, 2, 39]).

An alternative is to not solve directly for the EM fields themselves, but instead to initially solve a well conditioned equation for a quantity which is continuous across interfaces. Subsequently, the EM fields are obtained through a spatial differentiation with the field discontinuities defined by the property jumps. One such quantity is a vector potential with the Lorenz gauge, also called the Schelkunoff potential [37, 8, 33, 9], which we examine in this paper. In general, this potential has both scalar and vector components, and there are both electric and magnetic versions. Using the Lorenz gauge, the scalar potential can be expressed as a function of the vector potential, and as a result only the vector potential is needed to represent the EM field.

In this paper, we show that the Lorenz gauged vector potential representation exists for any member of $\mathcal{H}(\nabla\times)$. Thus one can use it to represent the electric field E as well as the magnetic field H. We prove that this representation exists for any frequency $\omega > 0$, if the permittivity ϵ is bounded and the magnetic permeability μ and the conductivity σ are bounded away from 0 and ∞. The electromagnetic properties ϵ, μ, σ are allowed to be discontinuous. We discuss an application of this potential for FEM approximation of the EM field. In principle, it is enough to use only the vector Lorenz gauged potential to represent the EM field. However, when the conductivity σ is not constant and the electric field is not divergence-free, it is difficult to find a weak equation involving only the vector potential. In particular, we show that the vector potential does not satisfy the weak form of the Helmholtz equation, sometimes erroneously used as a basis for FEM simulation [33]. For the general case of non divergence-free EM fields, we propose a mixed formulation involving the scalar and vector potentials.

We consider also the case of representing the magnetic field using a vector potential with the Lorenz gauge. If the magnetic permeability μ is constant, the magnetic field is divergence-free and the vector potential coincides with the magnetic field. We show that the Lorenz gauge approach leads to a regularized weak equation for the magnetic field involving a divergence term, and as a result the equation does not suffer from the large near null space.

We show that sesquilinear forms of the equations for both magnetic vector potential and electric scalar-vector formulations remain coercive at low frequencies. It makes iterative solvers fast even if only standard vector multigrid preconditioners [35] are used. Another advantage is that the considered vector potential is a member of $\mathcal{H}(\nabla\times) \cap \mathcal{H}(\nabla\cdot)$. This allows to use nodal elements, which have more widely available implementations than edge elements. The edge elements, due to a discontinuity of the shape functions across elements boundaries, require post processing to get a value of a field at a specific point within an element. In geophysical applications, the domain is a convex polygon, so nodal discretization is dense in $\mathcal{H}_0(\nabla\times) \cap \mathcal{H}(\nabla\cdot)$ or in $\mathcal{H}(\nabla\times) \cap \mathcal{H}_0(\nabla\cdot)$ [13, 6].

Regularization of the curl-curl equation using a divergence term has been also suggested in [1, 13]. The current paper extends these ideas to the case of nonconstant, complex valued electromagnetic properties and non divergence-free fields.