J. Partial Diff. Eqs. **14**(2001), 356–364
©International Academic Publishers Vol.14, No.4

INEQUALITIES OF EIGENVALUES FOR UNIFORMLY ELLIPTIC OPERATORS WITH HIGHER ORDERS

Jia Gao Zhao Peibiao and Yang Xiaoping
(School of Science, Nanjing University of Science & Technology, Nanjing 210094, China)
(Received Mar. 5, 2001)

Abstract Let $\Omega \subset R^m (m \geq 2)$ be a bounded domain with piecewise smooth boundary $\partial\Omega$. Let t be positive integer with $t > 1$. We consider the eigenvalue problems about (1.1) and (1.2), and obtain Theorem 2.1 and Theorem 2.2, which are generalizations of the results in [1–2]. This kind of problem is interesting and significant both in theory of partial differential equations and in applications to mechanics and physics.

Key Words Uniformly elliptic operators with higher orders; eigenvalues; eigenfuctions.

1991 MR Subject Classification 35P1.5.
Chinese Library Classification O175.4.

1. Introduction

Let $\Omega \subset R^m (m \geq 2)$ be a bounded domain with piecewise smooth boundary $\partial\Omega$. Let t be positive integer with $t > 1$. In [1], the following eigenvalue problems were studied:

$$\begin{cases} (-\Delta)^t u = \lambda u, & x \in \Omega \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{t-1} u}{\partial \nu^{t-1}} = 0, & x \in \partial\Omega \end{cases}$$

In this paper, we consider the generalized eigenvalue problems

$$\begin{cases} (-1)^t \displaystyle\sum_{i_1, i_2, \cdots, i_t = 1}^{m} D_{i_1 i_2 \cdots i_t}(a_{i_1 i_2 \cdots i_t}(x) D_{i_1 i_2 \cdots i_t} u) = \lambda w(x) u, & x \in \Omega \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{t-1} u}{\partial \nu^{t-1}} = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

and

$$\begin{cases} \displaystyle\sum_{s=2}^{t} (-1)^s \sum_{i_1, i_2, \cdots, i_s = 1}^{m} D_{i_1 i_2 \cdots i_s}(a_{i_1 i_2 \cdots i_s}(x) D_{i_1 i_2 \cdots i_s} u) = \lambda w(x) u, & x \in \Omega \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{t-1} u}{\partial \nu^{t-1}} = 0, & x \in \partial\Omega \end{cases} \tag{1.2}$$

where ν is the unit outward normal to $\partial\Omega$, $w(x) \in C(\bar{\Omega})$, $a_{i_1 i_2 \cdots i_s}(x) \in C^s(\bar{\Omega})$ with

$$0 < \mu \leq a_{i_1 i_2 \cdots i_s}(x) \leq \eta, \quad 0 < \frac{1}{\xi} \leq w(x) \leq \delta \tag{1.3}$$

for $i_1, i_2, \cdots, i_s = 1, 2, \cdots, m$ and $s = 2, 3, \cdots, t$, and μ, η, ξ, δ are positive constants.

For (1.1) and (1.2), we obtain two inequalities about λ_{n+1} in terms of $\lambda_1, \lambda_2, \cdots, \lambda_n$. This kind of problem is interesting and significant both in theory of partial differential equations and in applications to mechanics and physics (see [3]).

For convenience, throughout the paper we use the notations

$$D_k = \frac{\partial}{\partial x_k}, k = 1, 2, \cdots, m, \quad \nabla = (D_1, D_2, \cdots, D_m), \quad \Delta = \nabla^2, \quad \int = \int_\Omega$$

$$D_{i(s)} = D_{i_1 i_2 \cdots i_s} = D_{i_1} D_{i_2} \cdots D_{i_s}, \quad D_{(i_e)} = D_{i_1 i_2 \cdots i_{e-1} i_{e+1} \cdots i_s},$$

$$D_{(i_e i_f)} = D_{i_1 i_2 \cdots i_{e-1} i_{e+1} \cdots i_{f-1} i_{f+1} \cdots i_s}$$

$$a_{i(s)}(x) = a_{i_1 i_2 \cdots i_s}(x), \quad A_{i(s)} = \max a_{i_1 i_2 \cdots i_s}(x), \quad B_{i(s)} = \max |\nabla a_{i_1 i_2 \cdots i_s}(x)|$$

$$\sum_{i(s)=1}^{m} = \sum_{i_1, i_2, \cdots, i_s = 1}^{m}, \quad \sum_{(i_e, i_f)=1}^{m} = \sum_{i_1, i_2, \cdots, i_{e-1}, i_{e+1}, \cdots, i_{f-1}, i_{f+1}, \cdots, i_s = 1}^{m}, s = 2, 3, \cdots, t$$

2. Main Results

Theorem 2.1 *Let $m \geq 2$ and $\lambda_i (i = 1, 2, \cdots, n+1)$ be the eigenvalues of* (1.1) *with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n+1}$. Then*

$$\lambda_{n+1} \leq \lambda_n + \frac{4\delta^2 \xi^3}{m^2 n^2} \left(t(2t + m - 2) A_{i(t)} \sum_{i=1}^{n} \left(\frac{\lambda_i}{\mu\xi}\right)^{\frac{t-1}{t}} + t(t-1) B_{i(t)} \sum_{i=1}^{n} \left(\frac{\lambda_i}{\mu\xi}\right)^{\frac{2t-3}{2t}} \right) \cdot \sum_{i=1}^{n} \left(\frac{\lambda_i}{\mu\xi}\right)^{\frac{1}{t}} \tag{2.1}$$

Corollary 2.1 *Let $m \geq 2$ and $\lambda_i (i = 1, 2, \cdots, n+1)$ be the eigenvalues of* (1.1) *with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n+1}$. Then*

$$\lambda_{n+1} \leq \lambda_n + \frac{4\delta^2 \xi^2 \lambda_n}{m^2 \mu} \left(t(2t + m - 2) A_{i(t)} + t(t-1) B_{i(t)} \left(\frac{\lambda_n}{\mu\xi}\right)^{-\frac{1}{2t}} \right)$$

Remark 2.1 Take $a_{i(t)} \equiv 1$ and $i_1 = i_2 = \cdots i_t = 1, 2, \cdots, m$ in (1.1). Then

$$\lambda_{n+1} \leq \lambda_n + \frac{4}{m^2 n^2} t(2t + m - 2) \left(\sum_{i=1}^{n} \lambda_i^{\frac{1}{t}}\right) \left(\sum_{i=1}^{n} \lambda_i^{\frac{t-1}{t}}\right) \tag{2.2}$$

and

$$\lambda_{n+1} \leq \frac{\lambda_n}{m^2} (m^2 + 4t(2t + m - 2)) \tag{2.3}$$

Inequalities (2.2) and (2.3) are just the results of Theorems 1 and 2 in [1]

Remark 2.2 Take $t = 2, m = 1$ in (1.1). Then

$$\lambda_{n+1} \leq \lambda_n + \frac{8\delta^2 \xi^3}{n^2} \left(3A \sum_{i=1}^{n} \left(\frac{\lambda_i}{\mu\xi}\right)^{\frac{1}{2}} + B \sum_{i=1}^{n} \left(\frac{\lambda_i}{\mu\xi}\right)^{\frac{1}{4}} \right) \sum_{i=1}^{n} \left(\frac{\lambda_i}{\mu\xi}\right)^{\frac{1}{2}} \tag{2.4}$$