J. Partial Diff. Eqs. **13**(2000), 361–383
©International Academic Publishers Vol.13, No.4

ON CONVERGENCE OF A SEQUENCE OF PARAMETERIZED CLOSED CONVEX SETS AND ITS APPLICATIONS

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(Received Feb. 2, 2000)

Abstract Some abstract results about convergences for a sequence of parameterized nonempty closed convex sets, in the Mosco sense as well as in the sense of the local gap have been proved. By using these results, the convergences of some sequences of closed convex sets by certain general concrete structures are discussed.
Key Words Parameterized closed convex sets; convergence in Mosco sense; double limits of sets sequences; integrodifferential operator; variational inequality systems of arbitrary order.
1991 MR Subject Classification 39B72, 26B25, 40A05, 46B99.
Chinese Library Classification O178, O174.13, O173.1, O177.2.

1. Introduction

In the study of the solutions for variational inequalities of arbitrary order (See, e.g. [1]), we need to deal with different structures of closed convex sets. Sometimes, these sets do not meet the necessary requirements for consideration. Therefore, some questions arise. Could we use a sequence of sets, which can meet the known necessary requirements, to approach the original one? Could it be approximated by using the double limit? How? Under what kind of conditions? In what sense? e.g.

$$I\!K_{m,n} = \left\{ v \in H^1(\Omega), \quad \int_{\Omega_n} v dx \geq b_m \right\} \tag{1.1}$$

where $\{\Omega_n\} \subset \Omega$ and $\{b_m\}$ are sequences of domains and constants respectively? If we have, meas $(\Omega_n \triangle \Omega_\infty) \to 0$ and $b_m \to b_\infty$ as $n, m \to \infty$, would $I\!K_{m,n} \to I\!K_\infty$, where

$$I\!K_\infty = \left\{ v \in H^1(\Omega), \quad \int_{\Omega_\infty} v dx \geq b_\infty \right\}?$$

How can we define the limit properly? Does the limit depend on the order of n and m?

In this paper, we try to given some results to answer these questions.

The theory of closed convex sets plays an important role in the study of variational inequality problems. In particular, for higher order variational inequalities, the structures of closed convex sets can be very complicated. More practical examples of specific closed convex sets can be found, in e.g., elasto-plastic theory [2,3], capillary problems with prescribed volume ([4]), the restricted mean curvature of a horizontal plate problem ([5]) and elsewhere.

A variational inequality problem of arbitrary order is defined as follows:

Let $\mathbb{B}$ be the vector-valued Banach space $\boldsymbol{W}^{M,p}(\Omega) = [W^{M,p}(\Omega)]^L$, where $p \geq 1, \Omega$ is a bounded $C^{M-1,1}$ domain in $\mathbb{R}^N$, $M, N, L \in \mathbb{N}$. Consider the following problem

$$u \in \mathbb{K}, \quad \langle Au, v - u \rangle \geq 0, \quad \forall v \in \mathbb{K} \tag{1.2}$$

where A is an operator of order $2M$, $\boldsymbol{u} = (u_1, \cdots, u_L)$ is unknown. $\mathbb{K}$ is a closed convex and nonempty subset of $\mathbb{B}$, which may have very complicated structures.

One structure for a closed convex set in a Banach space $\boldsymbol{W}$ is as follows

$$\mathbb{K} = \{v \in \boldsymbol{W}, \quad T(v - \Psi) \in X\} \tag{1.3}$$

where the following conditions are also required:

• (C1) X is a closed convex set in $\boldsymbol{V}$, containing at least the origin, where $\boldsymbol{V}$ is a Banach space;

• (C2) $T : \boldsymbol{W} \to \boldsymbol{V}$ is a continuous linear map;

• (C3) $\Psi \in \boldsymbol{W}$.

With (C1)–(C3), it is also easy to verify that $\mathbb{K}$ is closed and convex. In additional, $\mathbb{K}$ is nonempty since $\Psi \in \mathbb{K}$.

Examples

• a. In the problem (1.2), if $T = I$, where I is the identity operator, $X = \boldsymbol{V} = \boldsymbol{W} = \boldsymbol{W}_0^{M,p}(\Omega)$, then $\mathbb{K} = \boldsymbol{W}_0^{M,p}(\Omega)$. This is the case of a system of equations;

• b. The closed convex set for the obstacle problem is

$$\mathbb{K} = \left\{ v \in \mathbb{B} : \quad v \geq \Psi, \text{ a.e. in } \Omega, \quad \left.\frac{d^i v}{d\nu^i}\right|_{\partial\Omega} = 0, \quad i = 0, 1, \cdots, M-1 \right\}$$

Here, $T = I$ and $X = \bigcap_{i=1}^{L} \{x_i \geq 0\}$. This is just the case discussed in [6];

• c. In [5], the closed convex set considered is $\mathbb{K} = \{v \in H^2(\Omega) : \alpha \leq \Delta v \leq \beta\}$.

• d. In (1.1), $\boldsymbol{W} = H^1(\Omega), \boldsymbol{V} = \mathbb{R}, T_n = \int_{\Omega_n}, X_m = [b_m, \infty), \Psi = 0$.

More examples of this kind of closed convex sets can also be found in [1] and [7].

In [1], we used some convergent results to consider the regularity of the solution for a variational inequality of arbitrary order. Therefore, it is interesting to study the convergence of the closed convex sets with specific structure, as well as the relations among the solutions corresponding to the approximate convex sets.

The abstract theory for the convergence of closed convex sets in a Banach space was studied by Mosco in the 1960s (See [8,9]). After Mosco's pioneering work, there followed many studies on this kind of convergence (See [10] and its references). Without