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ON THE CAUCHY PROBLEM FOR A CLASS OF SEMILINEAR HYPERBOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS AND DATA

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Abstract In this paper we establish the existence of local solution to the Cauchy problem for a class of semilinear second order hyperbolic equations including degenerated type equations with discontinuous coefficients and data.

Key Words Hyperbolic equations; Cauchy problem; discontinuous coefficients and data.

Classification 35L15, 35L70.

1. Introduction

In this paper we are concerned with the Cauchy problem for a class of second order hyperbolic equations with discontinuous coefficients and data in the two dimensional case as follows

$$(P)\begin{cases} u_{tt} - a(y)u_{xx} - u_{yy} = f(u) \\ u\mid_{t=0} = 0,\ u_t\mid_{t=0} = \psi(x,y) \end{cases}$$

where $a \in L^\infty(R^1)$ and satisfies

$$c\min(|y|^\alpha, 1) \le a(y) \le b$$

a.e. in R^1 with positive constants c and b, α, and $f \in C^\infty(R^1)$ with $f(0) = 0$.

In the paper [1], Chen has established the existence of solution for the Riemannian problem when $a = 1$. After that, Chen and Fang treated the Cauchy problem for second order strictly hyperbolic equations with cross discontinuous data in the case of smooth coefficients in [2]. In the paper [3], the author gets the existence of solution for the Cauchy problem with nonsmooth data when α is a positive even integer. Moreover if we let a be piecewise-constant valued, these equations are similar to the equations that describe wave motion in the different media. The method we use here may be used to solve the problem of wave motion in different media.

2. Some Notations and Main Result

Firstly, we give some definitions as follows

Definition 1 $H^{s,s'}(R^2) = \{u \in L^2(R^2) | (1+|D_x|^2+|D_y|^2)^{\frac{s}{2}}(1+|D_x|^2)^{\frac{s'}{2}} u \in L^2(R^2)\}$ *and be equipped with the norm* $\|u\|_{s,s'} = \|(1+|D_x|^2+|D_y|^2)^{\frac{s}{2}}(1+|D_x|^2)^{\frac{s'}{2}} u\|_{L^2(R^2)}$.

Definition 2 $X^{0,s}(R^2) = \{u \in L^2(R^2) | a^{\frac{1}{2}}(1+|D_x|^2)^{\frac{s}{2}} u_x \in L^2, u_y \in H^{0,s}\}$ *and be equipped with the norm* $\|u\|_{X,0,s} = \|u\|_{L^2} + \|a^{\frac{1}{2}} u_x\|_{0,s} + \|u_y\|_{0,s}$.

It is obvious that these two function spaces are Hilbert spaces. Without confusion, we set $H^{s,s'} = H^{s,s'}(R^2)$ and $X^{0,s} = X^{0,s}(R^2)$ in the following discussion.

It is obvious that we have

Lemma 1 *$C_c^\infty(R^2)$ is dense in $X^{0,s}$ under the norm $\|\cdot\|_{X,0,s}$.*

Proof For any $u \in X^{0,s}$, we have by easy computation

$$\|J_\varepsilon^x u - u\|_{X,0,s} \to 0$$

as $\varepsilon \to 0$, where J_ε^x is a standard molifier in R^1 (see Schechter [4]).

Set $u_\varepsilon = J_\varepsilon^x u$, then $u_\varepsilon \in H^1(R^2)$.

So if we denote $u_{\varepsilon,\delta} = J_\delta^y(u_\varepsilon)$, then we have for any fixed $\varepsilon > 0$

$$\|u_{\varepsilon,\delta} - u_\varepsilon\|_{H^1} \to 0$$

as $\delta \to 0$. Thus the desired result follows immediately.

Then we have

Theorem 1 (Existence) *If $s > \dfrac{\alpha}{2(\alpha+2)}$ and $\psi \in H^{0,s}(R^2)$, then there exists a $T > 0$ and a function u defined in $[0,T] \times R^2$ such that u is a strong solution to the problem (P) and $u \in L^\infty([0,T], X^{0,s}(R^2))$, $u_t \in L^\infty([0,T], H^{0,s}(R^2))$. Moreover, $u \in C([0,T] \times R^2)$.*

Theorem 2 (Uniqueness) *If u and v solve the problem (P) in weak sense with $u, v \in L^\infty([0,T], X^{0,s}(R^2))$, $u_t, v_t \in L^\infty([0,T], H^{0,s}(R^2))$ for some $s > \dfrac{\alpha}{2(\alpha+2)}$, then $u = v$ in $[0,T] \times R^2$.*

3. A Class of Sobolev Inequalities

In this section we will establish a class of Sobolev inequalities.

Proposition 1 *If $s > \dfrac{\alpha}{2(\alpha+2)}$, then for any function $u \in X^{0,s}(R^2)$ we have*

$$|u|_{L^\infty(R^2)} \le C(\Omega) \|u\|_{X,0,s}$$

where the constant $C(\Omega)$ is independent of u. Moreover, $u \in C^0(R^2)$ and

$$\sup_{x \in R^2} |u(x+h) - u(x)| \to 0$$