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A MAXIMUM PRINCIPLE FOR ELLIPTIC AND PARABOLIC EQUATIONS WITH OBLIQUE DERIVATIVE BOUNDARY PROBLEMS*

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Abstract This paper prove a maximum principle for viscosity solutions of fully nonlinear, second order, uniformly elliptic and parabolic equations with oblique boundary value conditions.

Key Words Maximum principle; viscosity solution; fully nonlinear equations.
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In this note, we prove an Aleksandrov-Backlman-Pucci type maximum principle when the boundary conditions consist of oblique derivative conditions and Dirichlet conditions. We will show that the maximum principle holds if a large portion of the boundary has Dirichlet boundary conditions.

This kind of estimates are important for the oblique derivative probiems. The reason is the following. If we blow up a solution of the oblique derivative problem near the boundary, the blow-up solution satisfies the above mixed boundary value problem. We will investigate this in a forthcoming paper.

1. The Maximum Principle, Elliptic Case

Let Ω be a bounded domain in R^n. Let $\partial\Omega = \partial_d\Omega \cup \partial_n\Omega$. Assume $\partial_d\Omega$ is closed and $\partial_n\Omega$ is open with respect to the relative topology of $\partial\Omega$.

Consider a fully nonlinear elliptic operator

$$F(D^2u, Du, u, x) = 0 \tag{1}$$

in Ω, with uniformly elliptic condition

$$\lambda|P| \leq F(M+P, v, u, x) - F(M, v, u, x) \leq \Lambda|P| \tag{2}$$

for any positive definite matrix P, where λ, Λ are fixed positive constants.

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We say u is a solution of (1), always according to [1], namely in the sense of viscosity solutions.

We will first consider a special class of operators

$$F(D^2u, Du, u, x) = g(x,t) \tag{3}$$

with the condition

$$F(0, P, u, x) \equiv 0 \tag{4}$$

We need some terminology. Let $\Gamma(\Omega)$ be the convex hull of Ω, namely $\Gamma(\Omega) = \cap\{D | D \supset \Omega, \text{ convex}\}$.

Generalized Gauss map for any domain D

$$G : \partial D \to 2^{S^{n-1}} \text{ (the subset of } S^{n-1})$$

$$x \to \{\theta : \quad \theta \text{ is an outer unit normal of } \partial D \text{ at } x\}$$

Let ds be the normalized surface measure on S^{n-1}.

$$ds(S^{n-1}) = 1$$

For A, a subset of R^n, let $A \triangleright \{0\}$ be the cone with vertex 0 generated by A.

Now, we want to consider the following problem.

$$\begin{cases} F(D^2u, Du, u, x) = g(x) & \\ \dfrac{\partial u}{\partial n} \geq 0 & \text{on } \partial_n\Omega \\ u \geq 0 & \text{on } \partial_d\Omega \end{cases} \tag{5}$$

Let C_n be the co-cone of $\partial_n\Omega$ as follows

$$C_n = \{v \in S^{n-1} | G(\partial_n\Omega) \cdot v < 0\}$$

Clearly $C_n \triangleright \{0\}$ is a convex set.

Theorem 1 *Let u be a continuous solution of (5). Assume $ds(C_n) \geq \alpha$ for some $\alpha > 0$. Then*

$$\sup_{\Omega} u^- \leq C\Big(\int_{\Gamma(u)=u} |g^-|^n\Big)^{\frac{1}{n}} \tag{6}$$

where $\Gamma(u)$ is the convex hull of u

$$\Gamma(u) = \sup\{v(x) | v \leq 0 \ on\ \partial_d\Omega,\ v \leq u \ convex\}$$

and the constant C depends only on λ, Lambda, n and the diameter of the domain Ω.

Lemma *$\Gamma(u)$ is C^{11}_{loc}.*

Proof We refer this to [1].

Proof of Theorem 1