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AN ALMOST FOURTH ORDER PARAMETER-ROBUST NUMERICAL METHOD FOR A LINEAR SYSTEM OF $(M \geq 2)$ COUPLED SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEMS

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Abstract. We present a high order parameter-robust finite difference method for a linear system of $(M \geq 2)$ coupled singularly perturbed reaction-diffusion two point boundary value problems. The problem is discretized using a suitable combination of the fourth order compact difference scheme and the central difference scheme on a generalized Shishkin mesh. A high order decomposition of the exact solution into its regular and singular parts is constructed. The error analysis is given and the method is proved to have almost fourth order parameter robust convergence, in the maximum norm. Numerical experiments are conducted to demonstrate the theoretical results.

Key words. Parameter-robust convergence, System of coupled reaction-diffusion problem, Generalized-Shishkin mesh, Fourth order compact difference scheme, Central difference scheme.

1. Introduction

Consider the following system of $(M \geq 2)$ coupled singularly perturbed linear reaction-diffusion equations

$$\boldsymbol{T u} := -\boldsymbol{E u}'' + \boldsymbol{A u} = \boldsymbol{f},\ x \in\ \Omega = (0,1) \tag{1a}$$

subject to the boundary conditions

$$\boldsymbol{u}(0) = \mathbf{p},\ \boldsymbol{u}(1) = \mathbf{q}, \tag{1b}$$

where $\boldsymbol{E} = \text{diag}(\varepsilon, \dots, \varepsilon)$, with small parameter $0 < \varepsilon \ll 1$. Suppose that the matrix $\boldsymbol{A} : \overline{\Omega} \to \mathbb{R}^{M,M}$ and the vector valued function $\boldsymbol{f} : \overline{\Omega} \to \mathbb{R}^M$ are four times continuously differentiable on $\overline{\Omega}$. We assume that the coupling matrix $\boldsymbol{A} = (\, a_{ij}(x)\,)_{M\times M}$ satisfy the following positivity conditions at each $x \in \overline{\Omega}$

$$a_{ij}(x) \leq 0,\ i \neq j, \tag{2}$$

$$a_{ii}(x) > 0,\ i = 1, \dots, M, \tag{3a}$$

$$\sum_{j=1, j\neq i}^{M} \left\| \frac{a_{ij}}{a_{ii}} \right\|_{\overline{\Omega}} < 1,\ i = 1, \dots, M, \tag{3b}$$

where $\|.\|_{\overline{\Omega}}$ denotes the continuous maximum norm on $\overline{\Omega}$. It is well known that under these assumptions the problem (1) possesses unique solution $\boldsymbol{u} \in C^6(\overline{\Omega})^M$ and exhibits two layers of width $O(\sqrt{\varepsilon}\ln(1/\sqrt{\varepsilon}))$ at both ends of the domain. These types of system of equations appear in the modeling of various physical phenomenon, such as the turbulent interaction of waves and currents [30], predator-prey population dynamics [8] and investigation of diffusion processes complicated by chemical reactions in electro analytic chemistry [29].

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The use of classical numerical methods on uniform mesh for solving these problems may give rise to difficulties when the singular perturbation parameter ε is sufficiently small. This leads to the development of the numerical methods that are parameter-uniform/parameter-robust/uniformly convergent with respect to small parameter ε. There are two classes of parameter-robust numerical methods: fitted operator methods and fitted mesh methods. Such methods require the physical and the mathematical knowledge about the problem. In case of a fitted mesh method the accuracy is guaranteed for a fixed number of mesh points, irrespective of the magnitude of the perturbation parameter. To achieve this a class of non-equidistant meshes, dense in layers, are available in the literature, see [1],[26],[28],[31]. The construction of these meshes depends strongly on priori information of the solution and its derivatives. A wide class of parameter-robust numerical methods based on this approach are discussed in [5],[6],[13],[23], and the references therein.

The methods based on fitted meshes, particularly the Shishkin meshes [28] gained popularity because of their simplicity and applicability to more complicated problems in higher dimensions; see [5],[6],[13],[23]. Further, we refer readers to the review article [9] for the progress on the Shishkin meshes in the area of singular perturbation. The Shishkin mesh is preceded by the Bakhvalov mesh [1], which is somewhat more complicated. Shishkin meshes are piecewise equidistant and typically consist of two(three) equidistant parts on the basis of one(two) transition points: one(two) dense part(s) in the layer(s) and one coarse part outside the layer(s). Bakhvalov meshes are generated by a suitable mesh generating function which appropriately redistributes equidistantly spaced points, so that the mesh is dense in the layer(s) region(s). Combinations of these two meshes are developed in [13] and [27]. Bakhvalov meshes are generalized and simplified in [31] and some improvements of Shishkin meshes are considered in [32],[33]. One of the important modifications of Shishkin mesh called generalized Shishkin mesh is developed by Vulanović and used in establishing the high order parameter-robust convergence of numerical methods, see [32].

Although an extensive amount of literature is available for the numerical solution of (uncoupled) singularly perturbed reaction-diffusion problems, while only few papers deal with the numerical analysis of coupled system of singularly perturbed reaction-diffusion problems. Systems of singularly perturbed problems have been studied as back as Bakhvalov [1]. Shishkin [29] examined a system of two parabolic partial differential equations analogous to (1), posed on an infinite strip. For the system of two coupled reaction-diffusion equations, some parameter-robust numerical methods are designed and analyzed in [3],[4],[14],[15],[19],[22],[23].

It is natural to think about the parameter-robust numerical methods for systems of more than two singularly perturbed reaction-diffusion equations. Kellogg et al. [10] considered a system of singularly perturbed reaction-diffusion problems in two dimensions with the same perturbation parameter for all equations and proved that the standard finite difference method on piecewise-uniform Shishkin mesh is second-order accurate (up to logarithmic factor). Some parameter-robust numerical methods for solving problem (1) are analyzed in [7],[16], and the references therein. But in all the cases the order of convergence is atmost two. Nevertheless, first time in [3] a HODIE technique is used to derive a third order uniformly convergent numerical method for system of two reaction-diffusion equations. High order numerical methods are very convenient from numerical point of view; the reason is that these methods produce small errors with a low computational cost. The objective of the present paper is to construct an almost fourth order parameter-robust