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Conjugate Symmetric Complex Tight Wavelet Frames with Two Generators

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Abstract. Two algorithms for constructing a class of compactly supported conjugate symmetric complex tight wavelet frames $\Psi = \{\psi_1, \psi_2\}$ are derived. Firstly, a necessary and sufficient condition for constructing the conjugate symmetric complex tight wavelet frames is established. Secondly, based on a given conjugate symmetric low pass filter, a description of a family of complex wavelet frame solutions is provided when the low pass filter is of even length. When one wavelet is conjugate symmetric and the other is conjugate antisymmetric, the two wavelet filters can be obtained by matching the roots of associated polynomials. Finally, two examples are given to illustrate how to use our method to construct conjugate symmetric complex tight wavelet frames which have some vanishing moments.

AMS subject classifications: 42C15, 94A12

Key words: Complex tight wavelet frame, conjugate symmetry, vanishing moments.

1. Introduction

Recently, research on wavelets mainly concerns conventional real-valued wavelet bases and filter banks (see [1–8]). However, complex-valued wavelet bases also have been successfully studied and they offer a number of potential advantageous properties (see [9–14]). For example, in Radar and Sonar applications, the complex I/Q orthogonal signals can be processed with complex filter banks rather than processing the I/Q channels separately. Additionally, it is shown in [9–11] that the complex Daubechies wavelet can be symmetric and orthogonal, whereas the real-valued wavelet cannot. Another advantage of

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complex wavelets compared to real-valued wavelets is that they provide both magnitude and phase information. It is well known that linear phase is desirable for reasons of both computational complexity and image quality, which is usually important in image and audio sinal analysis (see [14]). As we know, real-valued symmetric wavelets and complex conjugate symmetric wavelets can have the linear phase (see [9]). The (conjugate) symmetry of linear phase filters leads to a lower complexity hardware implementation because the multiplies involving origin (conjugate) symmetric coefficient pairs can be combined. In the real-valued wavelets case, in order to achieve symmetry in a wavelet system, many generalizations of wavelet frames have been proposed and investigated in the literature (see [1–7]).

In this paper, we are interested in complex conjugate symmetric wavelet frames which have a linear phase. As a generalization of an orthonormal wavelet basis, a tight wavelet frame is an overcomplete wavelet system that preserves many desirable properties of an orthonormal wavelet basis. Based on the real wavelet frames, researchers study a class of complex tight wavelet frames with three conjugate symmetric generators and give an explicit parametric formula for the construction in [15]. Though by increasing the number of generators in a tight wavelet frame one has a great deal of freedom to construct them from refinable functions, in many applications, for various purposes such as computational and storage costs, one prefers a tight wavelet frame with as small as possible number of generators. This motivates us to consider the construction of complex tight wavelet frames with two conjugate (anti) symmetric generators. Furthermore, we give a criterion for the existence of the two generators and provide a description of a family of solutions when the low pass filter is of even length.

2. Construction of conjugate symmetric complex wavelet frames

Let us recall that a frame in a Hibert space $\mathscr{H}$ is a family of its elements $\{f_k\}_{k\in\mathbb{Z}}$ such that, for any $f \in \mathscr{H}$, $\exists\ 0 < A \le B < \infty$,

$$A \parallel f \parallel^2 \le \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \le B \parallel f \parallel^2,$$

where optimal A and B are called frame constants. If $A = B$, the frame is called *a tight frame*.

The frame $\{\psi_{s;j,k}\}_{s=1}^{n}$, where $\psi_{s;j,k}(x) = 2^{j/2}\psi_s(2^j x - k)$, $j, k \in \mathbb{Z}$, generated by translations and dilations of finite number of functions, is called *an affine* or *wavelet frame*. In this case, $\psi_1, \cdots, \psi_n$ are called the *generators* or *framelets*.

As in [1–3], following the multiresolution framework, we suppose that the refinable function and wavelets satisfy the refinement equation

$$\phi = 2\sum_{k\in\mathbb{Z}} h_0(k)\phi(2\cdot -k), \quad h_0(k) \in \mathbb{C}, \tag{2.1a}$$

$$\psi_s = 2\sum_{k\in\mathbb{Z}} h_s(k)\phi(2\cdot -k), \quad s = 1, 2, \cdots, n, \ \ h_s(k) \in \mathbb{C}. \tag{2.1b}$$