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On Spectral Approximations by Generalized Slepian Functions

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Abstract. We introduce a family of orthogonal functions, termed as generalized Slepian functions (GSFs), closely related to the time-frequency concentration problem on a unit disk in D. Slepian [19]. These functions form a complete orthogonal system in $L^2_{\varpi_\alpha}(-1,1)$ with $\varpi_\alpha(x)=(1-x)^\alpha$, $\alpha>-1$, and can be viewed as a generalization of the Jacobi polynomials with parameter $(\alpha,0)$. We present various analytic and asymptotic properties of GSFs, and study spectral approximations by such functions.

AMS subject classifications: 33E30, 33C47, 41A30, 65N35

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1. Introduction

The investigation of time-frequency concentration problem back to 1960s gives rises to some interesting special functions with attractive properties. The most significant ones (see a series of papers by Slepian et al. [16, 17, 20]) are known as the prolate spheriodal wave functions (PSWFs) or Slepian functions, which are bandlimited and mostly time-concentrated within a finite interval. This discovery has motivated many subsequential research works in various directions (see, e.g., [4, 5, 7, 10, 12, 14, 15, 22–24, 27]).

In a very recent work [25], we introduced a family of generalized PSWFs as the eigenfunctions of a singular Sturm-Liouville problem, and interestingly, they are also the eigenfunctions of an integral operator. This orthogonal system is complete in $L^2_{w_\alpha}(-1,1)$ with $w_\alpha(x)=(1-x^2)^\alpha, \alpha>-1$, and generalizes both the PSWFs (from order zero to order α) and the Gegenbauer polynomials (to a system with a bandwidth tuning parameter). However, this study could not cover the case when the weight function is nonsymmetric (i.e., the Jacobi weight function $w_{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$ with $\alpha\neq\beta$). Indeed, it seems implausible to generate an orthogonal system of functions which is simultaneously

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the eigenfunctions of a second-order differential operator and an integral operator with complex exponential kernel, based on the argument in [25].

In this paper, we explore such a generalization, but restrict our discussion to the non-symmetric Jacobi weight $\varpi_\alpha(x) = (1-x)^\alpha$ with $\alpha > -1$. More precisely, we define the orthogonal system as the eigenfunctions of a Sturm-Liouville problem (see (2.9) below), and show that they satisfy an integral equation which has a close relation with the time-frequency concentration problem over a unit disk studied in D. Slepian [19] (so we term this new family of orthogonal functions as generalized Slepian functions (GSFs)). We derive some analytic and asymptotic properties of the GSFs and their associated eigenvalues, and study spectral approximations of functions in $L^2_{\varpi_\alpha}(-1,1)$ using the GSFs as basis functions.

The paper is organized as follows. In Section 2, we define the GSFs and describe the algorithm for their evaluation. We present various properties in Section 3, and derive the spectral approximation results using the GSFs in Section 4, together with some numerical experiments to support the analysis.

2. Generalized Slepian functions

In this section, we define the generalized Slepian functions, and introduce an efficient algorithm for their numerical evaluation.

2.1. Jacobi polynomials

We first review some properties of the Jacobi polynomials (cf. [21]). For $\alpha,\beta > -1$, the Jacobi polynomials, denoted by $J_n^{(\alpha,\beta)}(x), x \in I := (-1,1)$, are orthogonal with respect to the weight function $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$; namely,

$$\int_I J_m^{(\alpha,\beta)}(x) J_n^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x)\, dx = 0, \qquad \text{if } m \neq n.$$

In this paper, we mainly use the Jacobi polynomials with $\beta = 0$, and particularly, denote $J_n^{(\alpha)}(x) := J_n^{(\alpha,0)}(x)$ and $\varpi_\alpha(x) = w_{\alpha,0}(x) = (1-x)^\alpha$. We further assume that they are normalized so that

$$\int_I J_m^{(\alpha)}(x) J_n^{(\alpha)}(x) \varpi_\alpha(x)\, dx = \delta_{mn}, \tag{2.1}$$

where δ_{mn} is the Kronecker delta symbol. The Jacobi polynomials $\{J_n^{(\alpha)}\}$ are the eigenfunctions of the Sturm-Liouville problem

$$\mathscr{L}_x^{(\alpha)}\big[J_n^{(\alpha)}\big] := -\varpi_{-\alpha}\partial_x\big((1-x^2)\varpi_\alpha \partial_x J_n^{(\alpha)}\big) = \gamma_n^{(\alpha)} J_n^{(\alpha)}, \qquad x \in I, \tag{2.2}$$

with the corresponding eigenvalues $\gamma_n^{(\alpha)} = n(n+\alpha+1)$. Hereafter, we use ∂_x to denote the ordinary derivative $\frac{d}{dx}$, and likewise for higher-order ordinary derivatives.