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Orthogonal Polynomials with Respect to Modified Jacobi Weight and Corresponding Quadrature Rules of Gaussian Type

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Abstract. In this paper we consider polynomials orthogonal with respect to the linear functional $\mathcal{L} : \mathcal{P} \to \mathbb{C}$, defined on the space of all algebraic polynomials $\mathcal{P}$ by

$$\mathcal{L}[p] = \int_{-1}^{1} p(x)(1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\exp(\mathrm{i}\zeta x)\mathrm{d}x,$$

where $\alpha,\beta > -1/2$ are real numbers such that $\ell = |\beta-\alpha|$ is a positive integer, and $\zeta \in \mathbb{R}\backslash\{0\}$. We prove the existence of such orthogonal polynomials for some pairs of α and ζ and for all nonnegative integers ℓ. For such orthogonal polynomials we derive three-term recurrence relations and also some differential-difference relations. For such orthogonal polynomials the corresponding quadrature rules of Gaussian type are considered. Also, some numerical examples are included.

AMS subject classifications: 33C47, 41A55, 65D30

Key words: Orthogonal polynomials, modified Jacobi weight function, recurrence relation, Gaussian quadrature rule.

1. Introduction

In this paper we continue investigation on orthogonality with respect to the exponential modification of classical weight functions, studied in [5–8]. Let us suppose that $\alpha,\beta > -1/2$ are real numbers such that $\ell = |\beta-\alpha|$ is a positive integer, and $\zeta \in \mathbb{R}\backslash\{0\}$. We are concerned with the following measure

$$\mathrm{d}\mu(x) = (1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\exp(\mathrm{i}\zeta x)\chi_{[-1,1]}(x)\,\mathrm{d}x \tag{1.1}$$

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supported on the interval $[-1,1]$. We investigate the question connected with the existence of a sequence of orthogonal polynomials $\{\pi_n\}_{n\in\mathbb{N}_0}$ with respect to the linear moment functional $\mathcal{L}:\mathcal{P}\to\mathbb{C}$, defined on the space of all algebraic polynomials $\mathcal{P}$ by

$$\mathcal{L}[p]=\int_{-1}^{1} p(x)\,\mathrm{d}\mu(x)=\int_{-1}^{1} p(x)\,(1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\exp(\mathrm{i}\zeta x)\,\mathrm{d}x. \tag{1.2}$$

The corresponding moments are $\mu_k=\mathcal{L}[x^k]$, $k\in\mathbb{N}_0$.

This paper is organized as follows. In Section 2 the existence of orthogonal polynomials for some parameters α and ζ and for all positive integers $\ell=|\beta-\alpha|$ is proved. Section 3 is devoted to three-term recurrence relations as well as to some differential-difference relations. Finally, in Section 4 the corresponding quadrature rules of Gaussian type are considered. Such quadrature rules are suitable for computation of integrals of highly oscillatory functions of the form $\int_{-1}^{1} f(x)(1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x$. Notice that such kind of integrals appears in many branches of applied and computational science, e.g., for determining of the retarded potentials of electromagnetic field of a linear wire antenna (see [9]).

2. Existence of orthogonal polynomials

The measure (1.1) can be written in the following form

$$\mathrm{d}\mu(x)=\begin{cases}(1+x)^{\ell}(1-x^2)^{\alpha-1/2}\exp(\mathrm{i}\zeta x)\chi_{[-1,1]}(x)\,\mathrm{d}x, & \beta>\alpha,\\ (1-x)^{\ell}(1-x^2)^{\beta-1/2}\exp(\mathrm{i}\zeta x)\chi_{[-1,1]}(x)\,\mathrm{d}x, & \alpha>\beta.\end{cases}$$

Therefore, in the sequel we consider the measures

$$\mathrm{d}\mu^{\pm}(x)=(1\pm x)^{\ell}(1-x^2)^{\alpha-1/2}\exp(\mathrm{i}\zeta x)\chi_{[-1,1]}(x)\,\mathrm{d}x,$$

where $\alpha>-1/2$ and ℓ is a positive integer, i.e., we consider the existence of polynomials orthogonal with respect to the linear functionals

$$\mathcal{L}^{\pm,\zeta,\alpha,\ell}(p):=\mathcal{L}^{\pm}(p)=\int_{-1}^{1} p\,\mathrm{d}\mu^{\pm},\qquad p\in\mathcal{P}. \tag{2.1}$$

The moments

$$\mu_k^{\pm}=\int_{-1}^{1} x^k(1\pm x)^{\ell}(1-x^2)^{\alpha-1/2}\exp(\mathrm{i}\zeta x)\,\mathrm{d}x \tag{2.2}$$

can be expressed in terms of Bessel functions J_ν of the order ν (see [10, p. 40]). We restrict our attention only to the case $\zeta>0$, since the corresponding results for $\zeta<0$ can be obtained by a simple conjugation, because $\mu_k^{\pm}(-\zeta)=\overline{\mu_k^{\pm}(\zeta)}$, $k\in\mathbb{N}_0$.