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Immersed Interface Finite Element Methods for Elasticity Interface Problems with Non-Homogeneous Jump Conditions

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Abstract. In this paper, a class of new immersed interface finite element methods (IIFEM) is developed to solve elasticity interface problems with homogeneous and non-homogeneous jump conditions in two dimensions. Simple non-body-fitted meshes are used. For homogeneous jump conditions, both non-conforming and conforming basis functions are constructed in such a way that they satisfy the natural jump conditions. For non-homogeneous jump conditions, a pair of functions that satisfy the same non-homogeneous jump conditions are constructed using a level-set representation of the interface. With such a pair of functions, the discontinuities across the interface in the solution and flux are removed; and an equivalent elasticity interface problem with homogeneous jump conditions is formulated. Numerical examples are presented to demonstrate that such methods have second order convergence.

AMS subject classifications: 65N30

Key words: Immersed interface finite element methods, elasticity interface problems, singularity removal, homogeneous and non-homogeneous jump conditions, level-set function.

1. Introduction

In this paper, we consider the elasticity interface problem

$$\nabla \cdot \sigma + F = 0, \qquad \text{in } \Omega^- \cup \Omega^+, \tag{1.1}$$

where σ is the stress tensor, a 2×2 symmetric matrix, $F = (f_1,\ f_2)^T$ is a known body force. The domain Ω consists of Ω^- and Ω^+, $\Omega^- \cap \Omega^+ = \emptyset$, see Fig. 1 for an illustration. We assume that the interface $\Gamma = \overline{\Omega^-} \cap \overline{\Omega^+}$ separates Ω^- and Ω^+ is smooth enough (C^2). We also denote by $\mathbf{n}$ the unit vector normal to Γ pointing from Ω^- to Ω^+.

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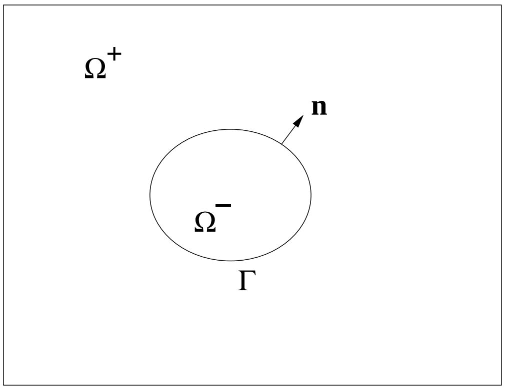

Figure 1: A diagram of the geometry of an elliptic interface problem.

For linearly elastic problems with small displacements, the relation between stress tensor and deformation is given by

$$\sigma_{ij} = \lambda(\nabla \cdot u)\,\delta_{ij} + 2\mu\,\varepsilon_{ij}(u), \tag{1.2}$$

where λ and μ are Lamé constants, $u = (u_1, u_2)^T$ is the displacement vector. The equations (1.1) can be written as the component form,

$$-\left\{(\lambda+2\mu)\frac{\partial^2 u_1}{\partial x_1^2} + (\lambda+\mu)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \mu\frac{\partial^2 u_1}{\partial x_2^2}\right\} = f_1, \qquad u_1\Big|_{\partial\Omega} = g_1, \tag{1.3}$$

$$-\left\{(\lambda+2\mu)\frac{\partial^2 u_2}{\partial x_2^2} + (\lambda+\mu)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \mu\frac{\partial^2 u_2}{\partial x_1^2}\right\} = f_2, \qquad u_2\Big|_{\partial\Omega} = g_2. \tag{1.4}$$

Due to the discontinuities in the coefficients, or/and source distribution along the interface Γ, the solution and flux are often discontinuous. The jump conditions across Γ can be written as

$$[u_1]_\Gamma = w_1, \tag{1.5}$$

$$[u_2]_\Gamma = w_2, \tag{1.6}$$

$$\left[\lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)n_1 + 2\mu\frac{\partial u_1}{\partial x_1}n_1 + \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)n_2\right]_\Gamma = q_1, \tag{1.7}$$

$$\left[\lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)n_2 + 2\mu\frac{\partial u_2}{\partial x_2}n_2 + \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)n_1\right]_\Gamma = q_2. \tag{1.8}$$

The jump conditions are called *natural* if

$$w_1 = w_2 = q_1 = q_2 = 0.$$