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The State Equations Methods for Stochastic Control Problems

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Abstract. The state equations of stochastic control problems, which are controlled stochastic differential equations, are proposed to be discretized by the weak midpoint rule and predictor-corrector methods for the Markov chain approximation approach. Local consistency of the methods are proved. Numerical tests on a simplified Merton's portfolio model show better simulation to feedback control rules by these two methods, as compared with the weak Euler-Maruyama discretisation used by Krawczyk. This suggests a new approach of improving accuracy of approximating Markov chains for stochastic control problems.

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1. Introduction

Many applications in finance and economics, such as the portfolio management problem, can be formulated as continuous time and continuous state stochastic control problems. These problems consists of the minimization of the cost function (in the finite horizon case)

$$J(\tau,x;u)=\mathbf{E}\left(\int_\tau^T L(x(t),u(x(t)),t)dt+s(x(T))\middle|x(\tau)=x\right) \tag{1.1}$$

by choosing the optimal Markov feedback control policy $\hat{u}(x)\in\mathscr{U}(x)\subset\mathbb{R}^m$, for all states $x\in\mathscr{R}\subset\mathbb{R}^n$, subject to the state equation

$$dx(t)=f(x(t),u(x(t)),t)dt+g(x(t),u(x(t)),t)dW(t), \tag{1.2}$$

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which is a controlled stochastic differential equation (CSDE). Here $f:\mathbb{R}^n\times\mathbb{R}^m\times[0,+\infty]\mapsto\mathbb{R}^n$, $g:\mathbb{R}^n\times\mathbb{R}^m\times[0,+\infty]\mapsto\mathbb{R}^{n\times d}$ are continuous functions and satisfy conditions that are sufficient to have a solution for (1.2), and $W(t)$ is a d-dimensional Brownian motion. $\mathscr{U}(x)$ is called the set of admissible controls given x, and $\mathscr{R}$ is the state space. The optimized cost function, denoted by $\hat{J}(\tau,x)$, is also called the optimal value function. That is,

$$\hat{J}(\tau,x)=\min_{u\in\mathscr{U}}J(\tau,x;u)=J(\tau,x;\hat{u}).$$

Except for some simple cases, e.g., those in [3, 4, 10], explicit solutions of continuous-time, continuous-state stochastic optimal control problems are very rare, and therefore numerical methods arise. The Markov chain approximation method is an efficient numerical approach which is widely used in financial economics. It goes back to Kushner (1977) and is described in Kushner and Dupuis [9]. The basic idea is to approximate the continuous-time, continuous-state problem by a discrete-time, discrete-state Markov chain model.

Krawczyk [7] proposed a totally different approach of finding transition probabilities based on discretisation of state equations, which is quite simple, intuitive and easy to understand. Effectiveness of his method is shown by [2, 7, 8]. The method of Krawczyk arouses our interest of investigating other numerical discretisations of the CSDE (1.2), as well as observing and comparing their effects on the final choice of the optimal control policy. As an example, we apply the midpoint rule instead of the Euler-Maruyama method to (1.2). The predictor-corrector methods (p-c^k) is also proposed in case of, e.g., nonlinear f or g.

Section 2 introduces the method of Krawczyk. The midpoint rule, p-c^k methods, and the Stratonovich stochastic differential equations, with which the midpoint rule is consistent, are stated in Section 3, where local consistency of Markov chains arising from the two kinds of methods are proved. Section 4 shows application of the two methods to a simplified Merton's portfolio model [7], and numerical experiments are performed in Section 5. Section 6 is a brief conclusion.

2. The method by Krawczyk

In the Markov chain approximation method, the state $x(t)$ is approximated by the Markov chain

$$\xi^h=\{\xi^h_k|k\in\mathbb{N}_0\}\subset\mathscr{R}_h, \tag{2.1}$$

where $\mathscr{R}_h$ is the discrete state space, and $h=(h_1,\cdots,h_n)\in\mathbb{R}^n_+$ is the state-step vector, and the cost function (1.1) is correspondingly changed to

$$J^{h,\Delta t^h}(\tau,x;u^h)=\mathbf{E}\left(\sum_{k=0}^{N-1}L(\xi^h_k,u^h_k,t^h_k)\Delta t^h_k+s(\xi^h_N)|\xi^h_0=x\right). \tag{2.2}$$

By choosing the Markov chain, its consistency with the state equation (1.2) is required. After the discrete model is constructed, the task then is to find the discrete optimal control $\hat{u}^h(x)$ for all $x\in\mathscr{R}_h$ to minimize the function $J^{h,\Delta t^h}$ in (2.2).