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Sparse Approximations of the Schur Complement for Parallel Algebraic Hybrid Solvers in $3D$[†]

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Abstract. In this paper we study the computational performance of variants of an algebraic additive Schwarz preconditioner for the Schur complement for the solution of large sparse linear systems. In earlier works, the local Schur complements were computed exactly using a sparse direct solver. The robustness of the preconditioner comes at the price of this memory and time intensive computation that is the main bottleneck of the approach for tackling huge problems. In this work we investigate the use of sparse approximation of the dense local Schur complements. These approximations are computed using a partial incomplete LU factorization. Such a numerical calculation is the core of the multi-level incomplete factorization such as the one implemented in pARMS. The numerical and computing performance of the new numerical scheme is illustrated on a set of large $3D$ convection-diffusion problems; preliminary experiments on linear systems arising from structural mechanics are also reported.

AMS subject classifications: 15A06, 15A23

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1. Introduction

The solution of partial differential equations (PDE) problems on large three dimensional ($3D$) meshes often leads to the solution of large sparse possibly unstructured linear systems. In this work, we mainly consider unsymmetric matrices resulting from the discretization of convection-diffusion type of problems. For their solution, we consider

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a parallel hybrid iterative-direct numerical technique. It is based on an algebraic preconditioner for the Schur complement system that classically appears in non-overlapping domain decomposition method. In earlier papers [3, 4], we studied the numerical and parallel scalability of this algebraic additive Schwarz preconditioners [2, 5, 6] where the preconditioner is built using exact local Schur complement matrices. This exact calculation is performed thanks to sparse direct solvers such as [1]. This calculation becomes prohibitive for large $3D$ problems both from a memory and computing time prospectives. To alleviate these costs while preserving its numerical robustness, we consider in this paper an approximation of the local Schur complement computed using a partial incomplete factorization following the approach implemented in the multi-level incomplete factorization schemes such as pARMS [7].

In Section 2, we describe the main components of the preconditioner that are the algebraic additive Schwarz approach and the variant of the dual thresholding $ILU(t,p)$ [10] enabling us to build the approximation of the local Schur complement. The memory and CPU-time benefits as well as the numerical and parallel behaviours are discussed in Section 4 through an extensive scalability study on large numbers of processors for model problems. More precisely, we mainly consider the $3D$ convection-diffusion problems defined by Eq. (1.1)

$$\begin{cases} -\epsilon \operatorname{div}(K.\nabla u) + v.\nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ϵ is a scalar, K a positive definite bounded tensor and v a velocity field defined on the computational domain Ω. Some preliminary experiments on linear systems arising from industrial structural mechanics computational are also reported.

2. The main components of the parallel preconditioner

Motivated by parallel distributed computing and the potential for coarse grain parallelism, considerable research activity developed around iterative domain decomposition schemes [8, 9, 13, 14]. The governing idea behind sub-structuring or Schur complement methods is to split the unknowns in two subsets. This induces the following block reordered linear system associated with the discretization of Eq. (1.1):

$$\begin{pmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} x_I \\ x_\Gamma \end{pmatrix} = \begin{pmatrix} b_I \\ b_\Gamma \end{pmatrix}, \tag{2.1}$$

where x_Γ contains all unknowns associated with sub-domain interfaces and x_I contains the remaining unknowns associated with sub-domain interiors. The matrix A_{II} is block diagonal where each block corresponds to a sub-domain interior. Eliminating x_I from the second block row of Eq. (2.1) leads to the reduced system

$$Sx_\Gamma = b_\Gamma - A_{\Gamma I}A_{II}^{-1}b_I, \quad \text{where } S = A_{\Gamma\Gamma} - A_{\Gamma I}A_{II}^{-1}A_{I\Gamma} \tag{2.2}$$

and S is referred to as the *Schur complement matrix*. This reformulation leads to a general strategy for solving (2.1). Specifically, an iterative method can be applied to (2.2). Once x_Γ is determined, x_I can be computed with one additional solve on the sub-domain interiors.