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Absorbing Boundary Conditions for Hyperbolic Systems

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Abstract. This paper deals with absorbing boundary conditions for hyperbolic systems in one and two space dimensions. We prove the strict well-posedness of the resulting initial boundary value problem in 1D. Afterwards we establish the GKS-stability of the corresponding Lax-Wendroff-type finite difference scheme. Hereby, we have to extend the classical proofs, since the (discretized) absorbing boundary conditions do not fit the standard form of boundary conditions for hyperbolic systems.

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Key words: Absorbing boundary conditions, hyperbolic system, Engquist and Majda approach, strict well-posedness, GKS-stability.

1. Introduction

This article is concerned with the numerical approximation of hyperbolic partial differential equations that are posed on an unbounded spatial domain (usually $\mathbb{R}^N$). When solving this *whole space problem* numerically one is facing the problem that one has to confine the computational domain. Typical examples for first order hyperbolic equations are the Maxwell's equations, the (linearized) Euler equations of fluid dynamics, the (linearized) shallow water equations and the classical hydrodynamic equation in semiconductor simulation [77] (without heat conduction term).

In some situations it is useful to apply a coordinate transformation with *conformal mappings* in order to transfer the original whole space problem to a new problem defined on a bounded domain. Unfortunately, the differential equation often becomes quite complicated [83] and moreover this transformation technique of conformal mappings fails, if the solution is oscillating at infinity and turns out to be not suitable for many physical problems [44]. The four more frequent numerical strategies to cope with this unbounded (or

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at least very large) spatial domain are infinite element methods (IEM), boundary element methods (BEM), absorbing layer approaches and artificial boundary conditions.

Here, in this work we restrict ourselves to the last strategy and comment briefly on the perfectly matched layer (PML) technique for hyperbolic systems. Hence, in the sequel, we confine the domain by introducing artificial boundaries without making any changes to the considered differential equation. At these artificial boundaries one defines so-called *absorbing boundary conditions* (ABCs), which are designed such that the solution of the bounded domain approximates well the solution on the original unbounded domain. In the literature the ABCs are also called transparent, open or nonreflecting boundary conditions. The quality of our approximation will be higher, if the components leaving the interior of the bounded domain (*outflow components*) induce only small reflections at the artificial boundary. Especially the amplitudes of the waves that are reflected from the artificial boundaries should be as small as possible [33]. The interested reader is referred to the review articles [10, 37, 51, 65, 95] and the references therein.

While this ABC approach is usually *PDE-based*: the ABCs are obtained by factorizing the underlying differential equation into outgoing and ingoing modes to minimize the reflections, the absorbing layer method can be considered as *material-based*: a damping (or lossy) medium is put around the domain of interest to damp (or even annihilate) the outgoing waves [52, 53]. In the first absorbing layer methods [68, 69] simply dissipative terms were added to the PDE in the layer. Later on, more advanced methods, e.g. [26] used grid stretching approaches. In a classical work [19] Bérenger proposed in 1994 the *perfectly matched layer* (PML) technique that possess a thinner layer and is (theoretically) reflectionless for waves of any incident angle and any wave number.

There exist a couple of applications of absorbing boundary conditions in the literature, e.g. in aeroacoustics [6], in quantum mechanics [10], in electro dynamics [2, 20], in fluid dynamics [17] and in geology [23]. In meteorology ABCs are used in local area weather forecasts [31], since the original domain (earth surface) would require a too high computational effort to solve the simulation in the given time frame and coarsening the grid would lead to unsatisfactory results.

This work consists of two parts: an analytic part and a numerical (discrete) part. In the first analytic section we will use the technique of pseudodifferential operators [84] to construct a *hierarchy of absorbing boundary conditions* for linear first order hyperbolic systems. Our procedure closely follows the classical work of Engquist and Majda [33]. Afterwards we will investigate the one-dimensional case and prove that the resulting initial boundary value problems (IBVP) are *well-posed* in the strict sense of Kreiss and Lorenz [71]. For hyperbolic systems in two spatial dimensions Engquist and Majda showed in 1977 that ABCs may give rise to not well-posed problems [33].

In the second numerical part of this article, the derived absorbing boundary conditions are discretized adequately and we show that the resulting *Lax-Wendroff difference scheme* for the IBVP in 1D is *GKS-stable* (stable in the sense of Gustafsson, Kreiss and Sundström). Let us stress the fact that the technique of our proof can be generalized to other finite difference schemes and other discretizations of the ABCs. We will present several numerical examples and focus on the numerical stability and the *discrete absorptions qualities* of the