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Preconditioned Iterative Methods for Algebraic Systems from Multiplicative Half-Quadratic Regularization Image Restorations

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Abstract. Image restoration is often solved by minimizing an energy function consisting of a data-fidelity term and a regularization term. A regularized convex term can usually preserve the image edges well in the restored image. In this paper, we consider a class of convex and edge-preserving regularization functions, i.e., multiplicative half-quadratic regularizations, and we use the Newton method to solve the correspondingly reduced systems of nonlinear equations. At each Newton iterate, the preconditioned conjugate gradient method, incorporated with a constraint preconditioner, is employed to solve the structured Newton equation that has a symmetric positive definite coefficient matrix. The eigenvalue bounds of the preconditioned matrix are deliberately derived, which can be used to estimate the convergence speed of the preconditioned conjugate gradient method. We use experimental results to demonstrate that this new approach is efficient, and the effect of image restoration is reasonably well.

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Key words: Edge-preserving, image restoration, multiplicative half-quadratic regularization, Newton method, preconditioned conjugate gradient method, constraint preconditioner, eigenvalue bounds.

1. Introduction

In image restoration, the restored image $\hat{\mathbf{x}} \in \mathscr{R}^p$ is estimated based upon a degraded data vector $\mathbf{b} \in \mathscr{R}^q$ by minimizing an energy function $\mathbf{J} : \mathscr{R}^p \to \mathscr{R}^q$, and the function $\mathbf{J}$

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consists of a data-fidelity term and a weighted regularization term Φ. Thus, it holds that

$$\hat{\mathbf{x}} = \min_{\mathbf{x}\in\mathscr{R}^p} \mathbf{J}(\mathbf{x}),$$
$$\mathbf{J}(\mathbf{x}) = \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2 + \beta\Phi(\mathbf{x}),$$

where $\beta > 0$ is a regularization parameter. The data-fidelity term given above assumes that $\mathbf{b}$ and $\mathbf{x}$ satisfy an approximate linear relation $\mathbf{A}\mathbf{x} \approx \mathbf{b}$, but that $\mathbf{b}$ is contaminated by noise. The treatment of using such a data-fidelity term is popular in computations of many inverse problems such as seismic imaging, non-destructive evaluation, and x-ray tomography; see, e.g., [5,8]. Here, we consider regularization terms Φ of the form

$$\Phi(\mathbf{x}) = \sum_{i=1}^{r} \phi(\mathbf{g}_i^T\mathbf{x}), \tag{1.1}$$

where $\mathbf{g}_i^T : \mathscr{R}^p \to \mathscr{R}$, $i = 1,\cdots,r$, are linear operators. Typically, $\{\mathbf{g}_i^T\mathbf{x}\}_{i=1}^r$ are the first- or the second-order differences between neighboring samples in $\mathbf{x}$. For example, if $\mathbf{x}$ is a one-dimensional signal, then it usually holds that $\mathbf{g}_i^T\mathbf{x} = x_i - x_{i+1}$, $i = 1,\cdots,p-1$, where $\mathbf{x} = (x_1, x_2, \cdots, x_p)^T \in \mathscr{R}^p$. Let $\mathbf{G}$ denote the $r\times p$ matrix whose ith row is $\mathbf{g}_i^T$, $i = 1,\cdots,r$, such that

$$\phi \not\equiv 0 \quad \text{and} \quad \ker(\mathbf{A}^T\mathbf{A}) \cap \ker(\mathbf{G}^T\mathbf{G}) = \{\mathbf{0}\}, \tag{1.2}$$

where $\ker(\cdot)$ denotes the kernel space of the corresponding matrix.

In this paper, we will focus on convex, edge-preserving potential functions $\phi : \mathscr{R} \to \mathscr{R}$ employed in (1.1), because they give rise to image and signal estimates of high quality involving edges and homogeneous regions. Examples of such functions (see, e.g., [5,13]) are listed as follows:

$$\phi_1(t) = |t| - \alpha\log(1 + |t|/\alpha), \tag{1.3}$$
$$\phi_2(t) = \sqrt{\alpha + t^2} - \sqrt{\alpha}, \tag{1.4}$$
$$\phi_3(t) = \log(\cosh(\alpha t))/\alpha, \tag{1.5}$$
$$\phi_4(t) = \begin{cases} t^2/(2\alpha), & \text{if } |t| \le \alpha, \\ |t| - \alpha/2, & \text{if } |t| > \alpha, \end{cases} \tag{1.6}$$

where $\alpha > 0$ is a prescribed parameter. In general, we assume that ϕ is convex, even, $\mathscr{C}^2$, and satisfies

$$\mathbf{A}^T\mathbf{A} \text{ is invertible} \quad \text{and/or} \quad \phi''(t) > 0, \quad \forall t \in \mathscr{R}, \tag{1.7}$$

where $\phi''(t)$ denotes the second-order derivative of the function $\phi(t)$ with respect to t. It is easy to see that the assumptions in (1.7) and (1.2) guarantee that, for every $\mathbf{y} \in \mathscr{R}^p$, the function $\mathbf{J}$ has a unique global minimum point.

However, the minimizers $\hat{\mathbf{x}}$ of the cost functions $\mathbf{J}$ involving edge-preserving regularization terms are nonlinear with respect to $\mathbf{x}$. Hence, their computations are quite complicated and costly. To simplify such computations, a multiplicative half-quadratic reformulation of