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# Preservation of Linear Constraints in Approximation of Tensors

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**Abstract.** For an arbitrary tensor (multi-index array) with linear constraints at each direction, it is proved that the factors of any minimal canonical tensor approximation to this tensor satisfy the same linear constraints for the corresponding directions.

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**Key words**: Multi-index arrays, tensors, linear constraints, low rank approximation, canonical tensor decomposition, multilevel matrices.

## 1. Introduction

Linear constraints define many important classes of structured matrices (Toeplitz, Hankel, various sparse matrices of special patterns etc.). A combination of Toeplitz and tensor structures was first considered in [3]. The common case of linear constraints along with tensor approximations of two-level matrices was first studied in [6]. Some estimates of tensor ranks were suggested in [2, 10, 11]. The interest to tensor approximations in combination with linear constraints is well justified by their role as a base for construction of fast algorithms in difficult cases, a good example is a superfast algorithm for approximate inversion of two-level Toeplitz matrices recently proposed in [7].

A matrix $A$ of order $n = p_1 p_2$ can be viewed as a matrix composed of blocks $a_{ij}$ of size $p_2 \times p_2$, where the indices $i, j$ run from 1 to $p = p_1$. In particular, $A$ can be of the form

$$A = A_r = \sum_{t=1}^{r} U_t \otimes V_t, \tag{1.1}$$

where $U_t$ and $V_t$ are matrices of order $p_1$ and $p_2$, respectively, and $\otimes$ denotes the tensor (Kronecker) product of matrices:

$$U \otimes V = \begin{bmatrix} u_{11}V & \cdots & u_{1p}V \\ \cdots & \cdots & \cdots \\ u_{p1}V & \cdots & u_{pp}V \end{bmatrix}.$$

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We denote by $\mathcal{T}_r = \mathcal{T}_r(p_1, p_2)$ the set of all matrices of the form (1.1) with real (for definiteness) entries for fixed values of $r$ and $p_1$, $p_2$, and we are especially interested in approximations

$$A \approx A_r \in \mathcal{T}_r$$

that minimize the Frobenius norm $||A - A_r||_F$ (the square root of the sum of squared entries in modulus). If $||A - B||_F > ||A - A_r||_F$ for all $B \in \mathcal{T}_k$ with $k < r$ and $||A - B||_F \geq ||A - A_r||_F$ for all $B \in \mathcal{T}_r$, then the minimizer matrix $A_r$ will be called the *minimal* approximation of tensor rank $r$.

Let us assume that the blocks $a_{ij}$ of a matrix $A$ satisfy linear constraints

$$\sum_{i=1}^{p} \sum_{j=1}^{p} c_{ij} a_{ij} = 0 \tag{1.2}$$

with some fixed scalar coefficients $c_{ij}$. For this case in [6] it was discovered and proved that the entries of each of the matrices $U_t$ of any minimal approximation $A_r$ are subject to the same constraints (1.2). It follows, for instance, that if $A$ is a block Toeplitz matrix (every block $a_{ij}$ is a function of $i - j$) then each of the matrices $U_t$ is a Toeplitz matrix. Similarly, if each of the blocks $a_{ij}$ is a Toeplitz matrix then each of the matrices $V_t$ ought to be Toeplitz.

In this paper we want to figure out to which extent the result of [6] can be generalized to the case of tensor approximations with an arbitrary fixed number of factors:

$$A_r = \sum_{t=1}^{r} U_t \otimes V_t \otimes \cdots \otimes W_t. \tag{1.3}$$

Let the number of factors in every summand be equal to $s$ and the orders of matrices $U_t, V_t, \cdots, W_t$ be $p = p_1, p_2, \cdots, p_s$, respectively. Then the order of $A_r$ is $n = p_1 p_2 \cdots p_s$. Denote the set of all matrices of the form (1.3) by $\mathcal{T}_r^s = \mathcal{T}_r^s(p_1, p_2, \cdots, p_s)$. For this case matrices $A_r$ are used as approximations for a given matrix $A$ of order $n = p_1 p_2 \cdots p_s$.

The matrix $A$ can be considered as a block matrix consisting of the blocks $a_{ij}$, $1 \leq i, j \leq p = p_1$. We will prove that from the viewpoint of preservation of linear constraints the case of arbitrary $s$ is analogous to the case $s = 2$: if the equations (1.2) are valid then the minimality of approximation implies that the same relationships (1.2) hold true for each of the matrices $U_t$.

One essential difference is still there. Suppose that $A \in \mathcal{T}_{r_{\max}}$ and $A \notin \mathcal{T}_r$ whenever $r < r_{\max}$. Then for $s = 2$ the minimal approximation is constructed via the singular value decomposition (SVD) for any $1 \leq r \leq r_{\max}$, whereas in the case $s > 2$ there could be some values $1 < r < r_{\max}$ for which a minimal approximation of tensor rank $r$ does not exist (cf. [5]). Moreover, there are no generalizations of the SVD to the case $s > 2$ that keep all the properties of this decomposition in the case $s = 2$ (some partial generalizations can be found in [1,9]), and therefore, some other techniques are needed.