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A Priori and A Posteriori Error Estimates of Streamline Diffusion Finite Element Method for Optimal Control Problem Governed by Convection Dominated Diffusion Equation

Ningning Yan* and Zhaojie Zhou

LSEC, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing 100080, China.

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Abstract. In this paper, we investigate a streamline diffusion finite element approximation scheme for the constrained optimal control problem governed by linear convection dominated diffusion equations. We prove the existence and uniqueness of the discretized scheme. Then a priori and a posteriori error estimates are derived for the state, the co-state and the control. Three numerical examples are presented to illustrate our theoretical results.

AMS subject classifications: 65N30

Key words: Constrained optimal control problem, convection dominated diffusion equation, streamline diffusion finite element method, a priori error estimate, a posteriori error estimate.

1. Introduction

Optimal control problem governed by advection dominated diffusion equations arises in many science and engineering applications (see., e.g., [9, 24, 30]). Recently, extensive research has been carried out on various theoretical aspects of the optimal control problems governed by advection diffusion and convection dominated equations, see, e.g., [2–4, 9, 27]. Most of them have been concerned with the unconstrained optimal control problem.

In this paper, we consider the following constrained optimal control problem:

$$\min_{u\in K\subset U}\{g(y)+j(u)\} \tag{1.1}$$

subject to

$$-\varepsilon\triangle y+\vec{b}\cdot\nabla y+ay=f+Bu \quad \text{in } \Omega, \tag{1.2a}$$
$$y=0 \quad \text{on } \partial\Omega, \tag{1.2b}$$

*Corresponding author. *Email addresses:* `ynn@amss.ac.cn` (N. Yan), `zzj534@amss.ac.cn` (Z. Zhou)

where the bounded open sets Ω, $\Omega_U \subset R^2$ with boundary $\partial\Omega$ and $\partial\Omega_U$,

$$K \subset U = L^2(\Omega_U)$$

is a bounded convex set, $g(\cdot)$ and $j(\cdot)$ are convex functionals. The details will be specified in the next section.

It is well known that the standard finite element discretizations applied to convection dominated diffusion problems lead to strongly oscillation. Some effective discrete schemes are investigated to improve the approximation properties of the standard Galerkin method and to reduce the oscillatory behavior, see, e.g., [11–13]. In [11], Hughes and Brooks first propose the streamline diffusion finite element method (also names SUPG), attracting more and more attentions because of its advantages of numerical stability and high order accurate. In [27], the authors apply SUPG method to the *unconstrained* optimal control problem governed by convection diffusion equation. They consider two approaches: optimize-then-discretize and discretize-then-optimize. A priori error estimates were proved for both the state, the co-state and the control. In [3], authors propose another stabilized finite element method for the discretization of the optimal control problem governed by convection diffusion equation with the constrained and unconstrained control.

In this paper, we apply the streamline diffusion finite element method to approximate the *constrained* optimal control problem (1.1)-(1.2a). We first derive the continuous optimality condition, which contains the state equation, the co-state equation and the optimal inequality. Then we use the streamline diffusion finite element methods to discretize the state equation and the co-state equation, and use the standard Galerkin method to approximate the optimal inequality directly. We prove the existence and the uniqueness for the approach. Moreover, a priori and a posteriori error estimates are obtained for both the state, the co-state and the control. The numerical examples are presented to illustrate our theoretical results.

Although a priori error estimates of *unconstrained* optimal control problem have been discussed in [27], there are new difficulties in our approach. Firstly, because the authors only consider the unconstrained problem in [27], the optimal inequality can be replaced by equality. Therefore the existence and the uniqueness of the problem become trivial and there is no need to prove them. While for the *constrained* problem, the existence and the uniqueness of our discrete SUPG scheme must be proved because it is not equivalent to a discrete optimal control problem. Moreover, the proof of a priori error estimate for the constrained control problems is more complicated than the one for the unconstrained control problems. Furthermore, we provide a posteriori error estimates of SUPG scheme for the optimal control problems, which is unavailable in literatures to our knowledge.

The outline of the paper is as follows: In Section 2, we present the streamline diffusion scheme for constrained optimal control problem governed by convection dominated diffusion equations. In Section 3, we prove the existence and uniqueness of the approach. In Sections 4 and 5, a priori and a posteriori error estimates are derived, respectively. In Section 6, we present three numerical examples to illustrate the theoretical results. In the last section, we briefly summarize the method used, results obtained and possible future extensions and challenges.