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A Posteriori Error Estimates of a Combined Mixed Finite Element and Discontinuous Galerkin Method for a Kind of Compressible Miscible Displacement Problems

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Abstract. A kind of compressible miscible displacement problems which include molecular diffusion and dispersion in porous media are investigated. The mixed finite element method is applied to the flow equation, and the transport one is solved by the symmetric interior penalty discontinuous Galerkin method. Based on a duality argument, employing projection estimates and approximation properties, a posteriori residual-type hp error estimates for the coupled system are presented, which is often used for guiding adaptivity. Comparing with the error analysis carried out by Yang (Int. J. Numer. Meth. Fluids, 65(7) (2011), pp. 781–797), the current work is more complicated and challenging.

AMS subject classifications: 65M12, 65M60

Key words: A posteriori error, discontinuous Galerkin method, compressible miscible displacement, mixed finite element, duality argument.

1 Introduction

We consider the following single-phase, miscible displacement problem of one compressible fluid by another in porous media:

$$d(c)\frac{\partial p}{\partial t}+\nabla\cdot\boldsymbol{u}=d(c)\frac{\partial p}{\partial t}-\nabla\cdot(a(c)\nabla p)=q,\qquad (x,t)\in\Omega\times J,\tag{1.1a}$$

$$\phi\frac{\partial c}{\partial t}+b(c)\frac{\partial p}{\partial t}+\boldsymbol{u}\cdot\nabla c-\nabla\cdot(\boldsymbol{D}(\boldsymbol{u})\nabla c)=(\widehat{c}-c)q,\qquad (x,t)\in\Omega\times J,\tag{1.1b}$$

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,\qquad (x,t)\in\partial\Omega\times J,\tag{1.1c}$$

$$\boldsymbol{D}(\boldsymbol{u})\nabla c\cdot\boldsymbol{n}=0,\qquad (x,t)\in\partial\Omega\times J,\tag{1.1d}$$

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$$p(x,0)=p_0(x), \qquad x\in\Omega, \tag{1.1e}$$
$$c(x,0)=c_0(x), \qquad x\in\Omega, \tag{1.1f}$$

where Ω is a polygonal and bounded domain in $\mathbb{R}^d$ $(d=2$ or $3)$ with the boundary $\partial\Omega$, $J=(0,T]$, $\boldsymbol{n}$ denotes the unit outward normal vector to $\partial\Omega$; $\boldsymbol{u}(x,t)$ represents the Darcy velocity of the mixture and $p(x,t)$ is the fluid pressure in the fluid mixture; $c(x,t)$ is the solvent concentration of interested species measured in amount of species per unit volume of the fluid mixture, $\phi(x)$ is the effective porosity of the medium and is bounded above and below by positive constants, $\boldsymbol{D}(\boldsymbol{u})$ denotes a diffusion or dispersion tensor which has contributions from molecular diffusion and mechanical dispersion. $\boldsymbol{D}(\boldsymbol{u})=d_m\boldsymbol{I}+|\boldsymbol{u}|\big(\alpha_l\boldsymbol{E}(\boldsymbol{u})+\alpha_t(\boldsymbol{I}-\boldsymbol{E}(\boldsymbol{u}))\big)$, where $\boldsymbol{E}(\boldsymbol{u})$ is the tensor that projects onto the $\boldsymbol{u}$ direction, whose (i,j) component is $(\boldsymbol{E}(\boldsymbol{u}))_{i,j}=u_iu_j/|\boldsymbol{u}|^2$; d_m is the molecular diffusivity and is assumed to be strictly positive; α_l and α_t are the longitudinal and transverse dispersion respectively, and are assumed to be nonnegative. The imposed external total flow rate q is a sum of sources and sinks. That is to say, $q=q^++q^-$, where $q^+=\max(q,0)$, $q^-=\min(q,0)$. The notation $\hat{c}$ denotes the specified c_w at source $(q>0)$ and the resident concentration at sinks $(q<0)$. It is supposed that $a(c)$, $b(c)$ and $d(c)$ are bounded.

Discontinuous Galerkin method (DG) belongs to a class of non-conforming methods (see [9, 12, 13, 15–19]) and they solve the differential equations by piecewise polynomial functions over a finite element space without any requirement on inter-element continuity-however, continuity on inter-element boundaries together with boundary conditions is weakly enforced through the bilinear form. DG methods are very attractive for practical numerical simulations because of their physical and numerical properties.

For the compressible miscible displacement problems, there are some literature about the DG approximations. In [4,6,7], a priori error for the compressible problem of dispersion-free $(\boldsymbol{D}(\boldsymbol{u})=d_m\boldsymbol{I})$ has been analysed. The authors have derived a priori error estimates of a discontinuous Galerkin approximation and a combined mixed finite element and discontinuous Galerkin method for a kind of compressible miscible displacement problems in [20, 21], respectively. But they only deal with a priori errors for the miscible displacement problem. Comparatively, the literature about a posteriori error for the miscible displacement problem is even scare. A posteriori error indicator is useful for adaptivity. A posteriori error of a discontinuous Galerkin scheme for the compressible miscible displacement problems with molecular diffusion and dispersion is presented in [22]. In this paper, a combined mixed finite element and symmetric interior penalty discontinuous Galerkin method is used to solve the completely compressible case with no restrictions on the diffusion/dispersion tensor. Based on a duality argument, employing projection estimates and approximation properties, a posteriori residual-type hp error estimates are obtained. Comparing with the error analysis of [22], the current work is more complicated and challenging.

The paper is organized as follows. In Section 2, we introduce a combined mixed finite element and discontinuous Galerkin method. Explicit a posteriori error estimates are presented in Section 3.