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ROBUST A-POSTERIORI ESTIMATORS FOR MULTILEVEL DISCRETIZATIONS OF REACTION–DIFFUSION SYSTEMS

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Abstract. We define a multilevel finite element discretization for a coupled stationary reaction–diffusion system in which each component can be defined on a separate grid. We prove convergence of the scheme and propose residual a-posteriori estimators for the error in the natural energy norm for the system. The estimators are robust in the coefficients of the system. We prove upper and lower bounds and illustrate the theory with numerical experiments.

Key words. multilevel finite elements, a-priori error estimates, a-posteriori error estimators, reaction-diffusion system

1. Introduction

In this paper we develop a-priori and a-posteriori analysis for finite-element discretizations of stationary reaction-diffusion systems. We are particularly interested in developing results which are uniform, in the sense made precise below, for families of such systems characterized by coefficients of different orders of magnitude. Additionally, since the individual components of such systems may have different variability, we recognize that they should be approximated on multilevel grids. The choice of such grids is guided by the a-posteriori error estimators.

A-posteriori analysis for finite element approximations of scalar self-adjoint elliptic equations is well developed [5, 35, 12]. The various error estimators that have been proposed differ in how closely they estimate the error and in the complexity of implementation and computations. In addition, their properties may depend significantly on the coefficients of the underlying problem.

Consider first the scalar stationary reaction–diffusion equation

$$-\nabla \cdot (a\nabla u) + \kappa u \quad = \quad f, \tag{1}$$

with a solution u. Consider also the corresponding standard Galerkin finite element formulation for (1) with a solution u_h, and an a-posteriori estimator η_s for the error $\mathcal{E}_s = \|\!| u - u_h |\!|\!|$ in the energy norm $\|\!| \cdot |\!|\!|$ associated with (1).

In general, the efficiency index $\theta_s := \frac{\eta_s}{\mathcal{E}_s}$ may significantly depend on the parameters in $\mathcal{P}_s = (a, \kappa)$. Standard theory, cf. [10, 5], considers $\mathcal{P}_s = \mathbf{1}^2 := (1, 1)$ and does not extend easily to the families of (1) where the parameters in $\mathcal{P}_s$ vary significantly. The concept of *robustness* [37, 36, 9, 39, 39, 40, 27, 26] allows to study such families of problems (1): the estimator η_s for (1) is *robust* if θ_s is uniform in $\mathcal{P}_s$, i.e., it remains constant or at least stable for a wide range of values in $\mathcal{P}_s$. Robust estimators are applicable, e.g., to singularly perturbed problems.

Now consider the problem of interest in this paper: the *system* of stationary reaction-diffusion equations posed in some domain Ω parametrized by $\mathcal{P} =$

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$(\lambda_1, \lambda_2, a, b, c)$

$$\lambda_1 u - \nabla \cdot (a \nabla u) + c(u - v) = f, \quad x \in \Omega, \tag{2}$$

$$\lambda_2 v - \nabla \cdot (b \nabla v) + c(v - u) = g, \quad x \in \Omega, \tag{3}$$

and complemented by appropriate boundary conditions.

The applications and analysis of general reaction-diffusion systems are considered, among other works, in [6, 33]. The special case of zero'th order coupling term associated with c, and of coefficients in $\mathcal{P}$ that may vary from case to case by orders of magnitude, has several applications. See for example the reaction-diffusion systems as in [23], double diffusion systems [7, 22, 31], ([30], II.5), and singular perturbations or regularizations of degenerate systems such as first-order reactions or adsorption at equilibrium and non-equilibrium [6, 25, 32]. See also pseudo-parabolic systems [24]. Our interest in this paper is in the numerical schemes; applications will be presented elsewhere.

An important observation is true for the families of solutions corresponding to the families of $\mathcal{P}$. In some applications the components u and v of the solution to (2)-(3) may have significantly different variability. In such cases it is natural to approximate the smooth component on a coarse grid and the less-smooth component on a fine grid. Such a multilevel discretization requires appropriate grid transfer operators so that the coupling term can be defined and the convergence ensured.

In addition, note that (2)-(3) can be seen as a prototype of a discretized-in-time parabolic system. While a-posteriori error estimation for parabolic problems can proceed along several paths [21, 34], some involve the consideration of robust estimates for (1) [3], and of the separation of spatial and temporal discretization errors without solving dual problems and/or backward heat equation [38, 8].

The above remarks motivate our work on robust estimators for the system (2)-(3). Our results i) extend the scalar estimators from [37] to the case of a coupled system, and ii) extend the work [4, 2] in which $\mathcal{P}$ was fixed. In addition, to our knowledge, ours is the only result concerning iii) multilevel schemes for (2)–(3).

A separate direction from a-posteriori error estimation is the use of special grids such as Shishkin and equidistributed meshes for resolving boundary layers in singularly perturbed problems [28, 19, 20]. For scalar problems (1) in 1D, it can be shown that with such grids, the dependence of the error of numerical solution on the parameters in $\mathcal{P}_s$ can be eliminated, e.g., by applying the MMPDE [19, 20]. We are unsure however how such grids can be constructed for systems when more than one of the parameters vary; it appears that the methods would not be a straightforward extension of [19].

The paper is organized as follows. We introduce notation and preliminaries in Section 2. In Section 3 we prove a-priori estimates for the multilevel discretization of (2)–(3). The main results of this paper are given in Section 4 where we define appropriate a-posteriori error estimators and prove upper and lower bounds; the estimators that we develop are robust in $\mathcal{P}$. Our theoretical results are illustrated by numerical experiments presented in Section 5.

We close with a few remarks on notation. Throughout the paper C means a generic positive constant; its value is different in each context in which it is used. The symbol $\partial_n w$ denotes the normal component of ∇w with respect to some boundary or edge. In all integrals we omit the symbol of integration variable; this helps to keep the expressions compact. Next, our theoretical results are given for $d = 2, 3$ spatial dimensions. The case $d = 1$ is also covered by the theory but the standard nomenclature and assumptions [14] do not apply; see [26] for robust estimates in $d = 1$.