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BARYCENTRIC COORDINATE BASED MIXED FINITE ELEMENTS ON QUADRILATERAL/HEXAHEDRAL MESH

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Abstract. This paper presents barycentric coordinate interpolation reformulated as bilinear and trilinear mixed finite elements on quadrilateral and hexahedral meshes. The new finite element space is a subspace of H(div). Barycentric coordinate interpolations of discrete vector field with node values are also known as the corner velocity interpolation. The benefit of this velocity interpolation is that it contains the constant vector fields (uniform flow). We provide edge based basis functions ensuring the same interpolation, and show how these basis functions perform as separate velocity elements.

Key Words. barycentric coordinate, corner velocity interpolation, mixed finite elements.

1. Introduction

The mixed finite element methods, e.g. [5] and streamline based simulations, [7], are commonly used in engineering applications like reservoir simulation. Consider the Raviart-Thomas or Raviart-Thomas-Nédélec conforming elements in 2 and 3 dimensions respectively denoted the $\mathcal{RT}_r$ and $\mathcal{RTN}_0$, cf. [22, 26, 17]. Because of complex reservoir geology, reliable flow simulations will in general require flexibility in the meshes. In streamline simulations extensions of the $\mathcal{RT}_0$-elements are used to interpolate the velocity field on irregular grids, see [6, 8, 9, 10, 13, 14, 15, 16, 20, 21]. However, the $\mathcal{RT}_r$- and $\mathcal{RTN}_0$-elements do not preserve constant vector fields (uniform flow) in their interpolation space, [16]. As a consequence they exhibit optimal-order convergence for the velocity in $H(\text{div})$, the velocity space, on general meshes, [2, 24].

To circumvent this problem and to improve the accuracy of streamline simulations, Hægland et al. devised the $\mathcal{CVI}$ (corner velocity interpolation) method [8] based on edge fluxes and bilinear or trilinear barycentric vertex interpolation. Barycentric coordinates $\phi_j(\boldsymbol{x})$ on a cell with vertex $\boldsymbol{x}_j$, $j = 1\ldots, n$ is the unique solution of $\sum \boldsymbol{x}_j\phi_j(\boldsymbol{x}) = \boldsymbol{x}$ and $\sum \phi = 1$, cf. [5] among many standard books. Over a triangle these are linear functions, scaled to be one or zero in the vertices. Barycentric coordinates offer linear precision, and used for interpolation of a velocity vector from the vertex values, this would clearly also reproduce a constant velocity field.

In this paper we take an alternative approach and propose a new multi-linear $H(\text{div})$ finite elements on quadrilateral and hexahedral meshes based on the idea of corner velocity interpolation. Hence, rather than constructing a velocity-interpolation that preserve the constant vectors based on given fluxes as a post processing step,

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we provide edge based shape functions for the $\mathcal{CVI}$. We then employ these functions as the velocity space in a mixed formulation. Some analysis of this velocity space is shown, and we provide a numerical study demonstrating properties of the new elements.

For completeness we review some of the main results on mixed finite elements related to convergence and reproduction of uniform flow on general meshes: Construction of robust $H(\mathrm{div})$-conforming mixed elements on deformed mesh, like quadrilaterals or hexahedra has been addressed by many authors over the last years, among many others see for instance [2, 3, 16, 24, 27] Construction of a finite element subspace of $H(\mathrm{div})$ relies on a multi-linear map to a unit reference cell. The *Piola transformation* relates the reference shape functions to the approximation space on the arbitrary irregular cell in physical space in a manner that preserves the fluxes. For general quadrilaterals, an analysis of vector fields defined via the Piola mapping by Arnold et al. [2], demonstrates a degradation of $H(\mathrm{div}, \Omega)$ convergence compared to rectangular meshes. The $\mathcal{RT}_0$ elements on shape-regular quadrilaterals, do in fact not converge in $H(\mathrm{div}, \Omega)$. For a vector field $\boldsymbol{v}$, the $L^2(\Omega)$ estimate is of optimal-order h, with some additional regularity requirements. In the $L^2(\Omega)$ estimate of the divergence of $\boldsymbol{v}$, on the other hand, accuracy of the interpolation is lost, and hence convergence is lost. This result was not fully resolved before the analysis given in [2].

However, in $L^2(\Omega)$, the $\mathcal{RT}_0$ elements retain optimal-order convergence for both the scalar and the vector fields [24]. Also, what usually is established is convergence in $H(\mathrm{div}, \Omega)$ for smooth grids, or for a sequence of h^2-uniform grids. I.e.; grids asymptotically reaching parallelogram meshes. The discrete fluxes from the $\mathcal{RT}$ element can immediately also be post-processed with an alternative cell interpolation, related to the Arnold-Boffi-Falk elements proposed in [2], to retain full $H(\mathrm{div})$ convergence on general rough meshes, c.f., [12]. Moreover, in [2] they show that for full $H(\mathrm{div})$ convergence of finite element soultions built on the Piola mapping, the lowest-order discrete space on $\mathcal{R}$ has to contain the constants. By exact representation of uniform flow, we ensure reproduction of constants on the physical space instead.

Similar results was obtained for the $\mathcal{RTN}_0$ space on hexahedra meshes in [3]. They prove convergence in $H(\mathrm{div}, \Omega)$ for shape-regular asymptotically parallelepiped meshes. Numerical experiments support this result by indicating no $H(\mathrm{div}, \Omega)$ convergence for meshes of trapezoidal shape. These meshes are conceptually similar to the trapezoidal meshes used in [2], which yield the same numerical results for 2D $\mathcal{RT}_0$ elements. The lack of convergence for the $\mathcal{RTN}_0$ space on general hexahedra meshes is also demonstrated in, e.g., [23]. This deficiency of the $\mathcal{RTN}_0$ velocity space, can be associated to the lack of reproduction of uniform flow. In [16], it is shown that, on a general hexahedron, a constant flow field does not imply linear face fluxes. Hence, the $\mathcal{RTN}_0$ velocity space obtained via the Piola mapping, which implicitly yields a linear flux reconstruction, does not contain the constant functions. The findings in [19] generalize this observation. They prove that for a general hexahedron with bilinear faces, both a local reconstruction of velocity based on the six face fluxes and exact representation of uniform flow (constant velocity field), can not be satisfied in $H(\mathrm{div}, \Omega)$.

The rest of this paper is organized as follows; In the next section we define the preliminaries. Then, in section 3, we present the new $\mathcal{CVI}$ finite element space, and furthermore in section 3.1, we find that the $\mathcal{CVI}$ space can be regarded as a perturbation of the $\mathcal{RT}_0$ elements. Section 3.2 provides the $\mathcal{CVI}$ shape functions for